Using Gravity Assist Maneuver to Increase Fuel Consumption Efficiency for a Space Probe Entering Jupiter's Orbit

Team 538

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Problem A. Gravitationally Assisted Jupiter Orbital Entry

Abstract

In this investigation of whether the gravity assist maneuver generates significant fuel savings for a space probe approaching Jupiter at 20 km/s, we calculate the amount of fuel needed the slow the probe down to enter Jupiter's orbit at an orbital radius of 1.2 times the radius of Jupiter. In both cases (with and without the gravity assist maneuver), the speed at the optimal point for the probe to release its thrusters (consume its fuel) is calculated, and then the difference between the result and the orbital speed required is computed. These results are then placed into Tsiolkovsky's Rocket Equation [4] to determine the amount of mass lost (fuel consumed) in each case. The findings show that without the gravity assist maneuver, the probe would need to spend 98.5% of its mass as fuel while it only needs to spend 96.8% of its mass as fuel with the gravity assist maneuver around Jupiter's moon Io. In a practical sense, this means that the probe can now carry twice the amount of non-fuel material to complete the same task by implementing this method. Thus we conclude that the gravity assist maneuver can produce significant fuel savings for the space probe.

1 Background

The gravity assist maneuver is a technique employed by a spacecraft to change its path and speed while saving propellant, and thus, expense. The concept is to use the relative motion and gravity of astronomical objects in order to accelerate/decelerate or redirect the spacecraft. The idea was first proposed by astronomical pioneers Yuri Kondratyuk and Friedrich Zander in their papers published in 1925 and 1938, respectively. It was first implemented in 1959 by the Soviet probe Luna 3 to photograph the far side of Earth's moon. Since then, notable successful implementations include space probes Mariner 10, Voyager 1, and Galileo. [2]

2 Introduction

In this paper, we compare the amount of fuel consumed by a space probe to enter Jupiter's orbit with and without implementing the gravity assist maneuver through Jupiter's moon Io. Given the initial speed of 20 km/s, we calculate the amount of fuel consumed to reduce the speed to the orbital speed required at 1.2 times the radius of Jupiter under the optimal trajectory. Then, using the same initial conditions, we calculate the amount of fuel needed to complete the same task but performing the gravity assist maneuver on Io first. Throughout the paper, we assume that the only astronomical objects whose gravitational fields can influence the space probe are Jupiter and Io. In the small time span to perform the gravity assist maneuver and the deceleration, we assume that the paths of Io's orbit around Jupiter and Jupiter's orbit around the sun are relatively linear. During the gravity assist maneuver, the effects of Jupiter's gravitational pull on the space probe is assumed to be negligible. The change in speed caused by fuel consumption is calculated in both cases and is used to compare the amounts of fuel used in the process to determine whether the gravity assist maneuver generates significant fuel savings for the probe.

3 The Tsiolkovsky Rocket Equation

We first wish to derive an expression for the amount of fuel needed to change the speed of the space probe by a certain amount Δv. Under the ideal conditions assumed where there are no external forces, this will result in the Tsiolkovsky rocket equation [4].

Since there are no external forces acting on the probe, by the Conservation of Momentum, the only way for it to change its motion is for it to throw away part of its own mass in the opposite direction. Let the initial mass of the probe be m_0 and the speed that it emits its exhaust be u, relative to the probe itself. If v is the speed of the rocket at some time, momentum will be mv. Assuming that the probe emits exhaust at a constant rate, after some interval of time Δt, the mass of the probe will be $m + \Delta m$ (Δm will be a negative value, as the mass is decreasing.) and the speed will be $v + \Delta v$. Now the exhaust will have mass

$-\Delta m$ (again, since Δm is a negative quantity), and it will be moving back at a speed $u - v$.

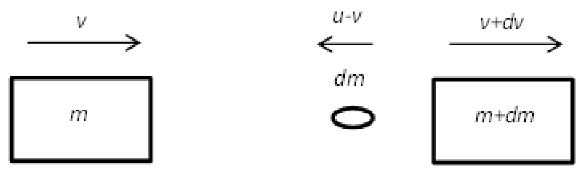

Figure 1: The space probe's momentum before and after emitting exhaust

By the Conservation of Momentum, we have

$$mv = (m + \Delta m)(v + \Delta v) + (-\Delta m)(v - u)$$

As Δt goes to 0, $\Delta m \to dm$ and $\Delta v \to dv$, and we can solve the equation.

$$mv = (m + dm)(v + dv) - (dm)(v - u)$$
$$mv = mv + mdv + vdm + dmdv - vdm + udm$$

The term $dmdv$ is the product of two differentials, so it is extremely small and negligible.

$$-mdv = udm$$
$$-\frac{dv}{u} = \frac{dm}{m}$$
$$\int_{v_0}^{v} -\frac{dv}{u} = \int_{m_0}^{m} \frac{dm}{m}$$
$$-(\frac{v}{u} - \frac{v_1}{u}) = \ln m - \ln m_0$$
$$\Delta v = u \ln \frac{m_0}{m} \tag{1}$$

4 Fuel Needed without Gravity Assist Maneuver

In the case without the gravity assist maneuver around a moon, the probe will fly toward Jupiter, and at some point when it is curving around Jupiter and its path is aligned with the desired circular orbit, the probe should apply thrust to slow itself to the appropriate speed. But at which point should it do this? We claim that we should aim the probe so that the point of closest approach is the orbital distance, and it is at this point that it should change its speed.

To understand this, consider the Conservation of Energy. The space probe has some kinetic energy as it is approaching Jupiter (but no potential energy, as it is far away), and it needs to reach a certain total energy to be in a particular orbit. (This will be negative since it is bound.) From the previous section, we know that the fuel needed (change in mass) to decrease the speed is exponentially proportional to the change in speed, but the change in energy is greatest for a given change in speed when the speed itself is the highest. This is when the probe is closest to Jupiter, since it will have the lowest potential energy and is when we should aim to change the speed.
One thing to note before we proceed with the calculations is that Jupiter itself is moving relative to the Sun, so the initial velocity of the rocket that we really want to consider is the velocity relative to Jupiter. This is given by

$$v_0 = \sqrt{v_p^2 + v_J^2} \tag{2}$$

since the two velocities are perpendicular.

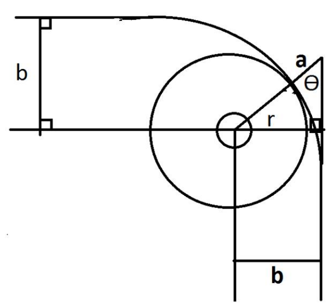

Figure 2: The space probe's trajectory around Jupiter

We can determine the speed of the probe when it reaches the point of closest approach using the Conservation of Energy.

$$\begin{aligned} E_1 &= E_2 \\ \frac{1}{2}mv_0^2 &= \frac{1}{2}mv^2 - \frac{GMm}{r} \end{aligned}$$

Let $\mu_J = GM$ be the standard gravitational parameter of Jupiter.

$$v = \sqrt{v_0^2 + \frac{2\mu_J}{r}} \tag{3}$$

To figure out the speed that we actually need here, we relate the gravitational force of Jupiter to the centripetal force.

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$
$$v = \sqrt{\frac{\mu_J}{r}} \tag{4}$$

Now we can calculate Δv and substitute values to get the amount of fuel needed using equation (1).

$$|\Delta v| = \sqrt{v_0^2 + \frac{2\mu_J}{r}} - \sqrt{\frac{\mu_J}{r}} = u \ln \frac{m_0}{m}$$
$$\frac{m}{m_0} = \exp\left(\frac{\sqrt{\frac{\mu_J}{r}} - \sqrt{v_0^2 + \frac{2\mu_J}{r}}}{u}\right)$$

Using the values $\mu_J = 1.266 \times 10^{17}$ N m^2 kg^{-1} [6], $r = 8.389 \times 10^7$ m [6] (1.2 times the radius of Jupiter), $v_J = 13.07$ km s^{-1} [6], $v_0 = \sqrt{v_p^2 + v_J^2} = 23.89$ km s^{-1}, and $u = 5000$ m s^{-1} (a very high speed for space probes) [1] we get that $\frac{m}{m_0} = 98.5\%$. This means that the space probe would only be able to keep 1.5% of its original mass after going into orbit.

4.1 Calculating Other Useful Quantities

In this situation with a probe trying to directly enter orbit, we can calculate several other important quantities that will be useful in the later discussion. These calculations will be valid for the probe entering a moon's gravitational field as well as Jupiter's so we use the more general μ instead of μ_J

First, we will calculate the impact parameter b that is needed to reach this situation. Using the Conservation of Angular Momentum, we get that

$$L_1 = L_2$$
$$mv_0 b = mvr$$
$$b = \frac{vr}{v_0}$$
$$b = \frac{r\sqrt{v_0^2 + \frac{2\mu}{r}}}{v_0}$$
$$b = \sqrt{r^2 + \frac{2\mu r}{v_0^2}} \tag{5}$$

Now we calculate the distance of closest approach given the initial conditions (speed and impact parameter), which will be needed in the next section. Substitute $v = \frac{v_0 b}{r}$ from the Conservation of Angular Momentum into the equation for the Conservation of Energy to get

$$\begin{aligned}\frac{1}{2}mv_0^2 &= \frac{1}{2}m\frac{v_0^2b^2}{r^2} - \frac{GMm}{r} \\ 0 &= v_0^2r^2 + 2\mu r - v_0^2b^2 \\ r &= \frac{-\mu + \sqrt{\mu^2 + v_0^4b^2}}{v_0^2} \qquad (6)\end{aligned}$$

5 Space Probe Passing Io

Here, we will examine the effects that Io has on the space probe as the probe passes by Io to perform the gravity assist maneuver.

From equation (6) of the previous section, the distance of closest approach is given by

$$r = \frac{-\mu + \sqrt{\mu^2 - v_0^4b^2}}{v_0^2}$$

Using the formula for hyperbolic excess velocity[3]:

$$\begin{aligned}v_0 &= \sqrt{\frac{\mu}{a}} \\ a &= \frac{\mu}{v_0^2}\end{aligned}$$

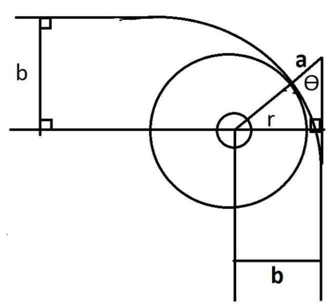

Figure 3: The space probe's trajectory around Io for the gravity assist maneuver

In the diagram, 2θ is the angle between the initial and final velocity vectors.

We now solve for θ:

$$\begin{aligned}
\theta &= \sin^{-1}\left(\frac{b}{a+r}\right) \\
&= \sin^{-1}\left(\frac{b}{\dfrac{\mu}{v_0^2}+\dfrac{-\mu+\sqrt{\mu^2-v_0^4b^2}}{v_0^2}}\right) \\
&= \sin^{-1}\left(\frac{bv_0^2}{\sqrt{\mu^2+v_0^4b^2}}\right) \\
&= \sin^{-1}\left(\frac{1}{\sqrt{\dfrac{\mu^2}{b^2v_0^4}+1}}\right)
\end{aligned} \tag{7}$$

6 Gravity Assist Maneuver around Io

We come to the main result of our discussion: whether or not performing a gravity assist maneuver around one of Jupiter's moons has a significant impact on the amount of fuel needed. We choose to use Jupiter's moon Io in our calculations because it is one of Jupiter's largest moons [9] so its gravitational influence can alter the probe's path more. Its orbit is very close to Jupiter, which means that it has a high speed relative to Jupiter compared with the other moons. This is advantageous, as we will see, since the effectiveness of a gravity assist maneuver depends largely on the speed of the object you are maneuvering around [2]. Additionally, the inclination angle of Io's orbit around Jupiter is only $2.21°$ to the ecliptic [5], so we can simplify the problem considerably by taking it as essentially two dimensional.

6.1 How the Maneuver Works

This idea of a gravity assist maneuver may at first seem to violate the Law of Conservation of Energy since it does not seem possible to gain or lose speed for free. However, this is because we have not considered the moon itself to be moving as well, so that before and after the maneuver, the space probe would have the same speed *relative to the moon.* The direction of motion of the probe changes, so that when the speed is then calculated relative to Jupiter, because of the way the new relative velocity vectors add up, the magnitude can be made greater or less than the original. [2]

The classic example of this is throwing a baseball against a wall. [2] Let's say the initial speed is 50 km/h. If the wall was stationary, the ball would just bounce back at the same speed that it came in at. Now imagine that you were behind a train moving at 20 km/h, and you threw the baseball at the train at the same speed. The train would see the ball coming in at only 30 km/h, so it would send it back at 30 km/h relative to itself. But this would be only 10 km/h relative to you, so we've slowed down the baseball by 40 km/h! If the train was instead coming toward you (I don't know why you would be standing in front of a train.), the ball would be moving 70 km/h relative to the train, the train would send it back at 70 km/h relative to itself, which would then be 90 km/h relative to you. Thus, in this case we've increased the baseball's speed by 40 km/h.
As you can see, this can be a very powerful technique to save fuel, but where does the excess energy go? It gets absorbed by the moon, but since the moon has so much mass compared to the space probe, the effect will not be noticeable at all. From this analogy, we see that to gain speed, we must be moving in the opposite direction of the moon at first and with it after, which means going behind it. To lose speed, we first move with the moon and against it after, which means going in front of it.

6.2 Details in the Case of Io

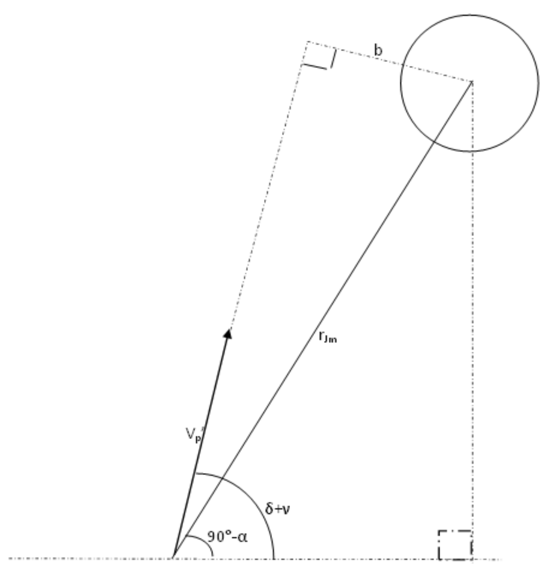

Figure 4: The position of Io relative to Jupiter

The space probe will be coming toward Jupiter at some speed $v_0 = \sqrt{v_p^2 + v_J^2}$

(equation (2)) relative to Jupiter (take the whole problem relative to Jupiter). At the point when the maneuver is done, the moon and Jupiter make an angle α with the probe's velocity toward Jupiter. Assume without loss of generality that Io is moving clockwise. (We can look at the situation upside down if it is counterclockwise.) This way, the probe will go to the left of Io to lose speed and Io is currently on the left side of Jupiter from the probe's perspective (see Figure 4). The distance from the moon to Jupiter is around 15 times the radius of Jupiter [7, 6], so we make the rough estimate of dividing the problem into two separate parts: first, the gravity assist maneuver to change the speed and direction of the probe; and second, the problem of needing to slow down further to the orbital speed of Jupiter at 1.2 times the radius, which is exactly the same problem as solved in section 4.

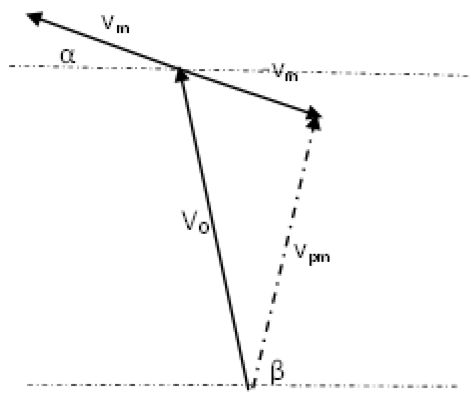

Figure 5: Velocity of the space probe relative to Io

Assume Io's orbit to be relatively circular so that its velocity makes an angle α with the horizontal. We are also given the probe's initial velocity, so we use trigonometry to calculate its velocity relative to the moon.

$$v_{pm} = \sqrt{v_0^2 + v_m^2 - 2v_0 v_m \cos(90^\circ - \alpha)}$$

$$v_{pm} = \sqrt{v_0^2 + v_m^2 - 2v_0 v_m \sin\alpha}$$

$$\frac{v_m}{\sin(90^\circ - \beta)} = \frac{v_{pm}}{\sin(90^\circ - \alpha)}$$

$$\beta = \cos^{-1}\left(\frac{v_m}{v_{pm}}\cos\alpha\right)$$

Now we look at the situation after it swings around the moon.

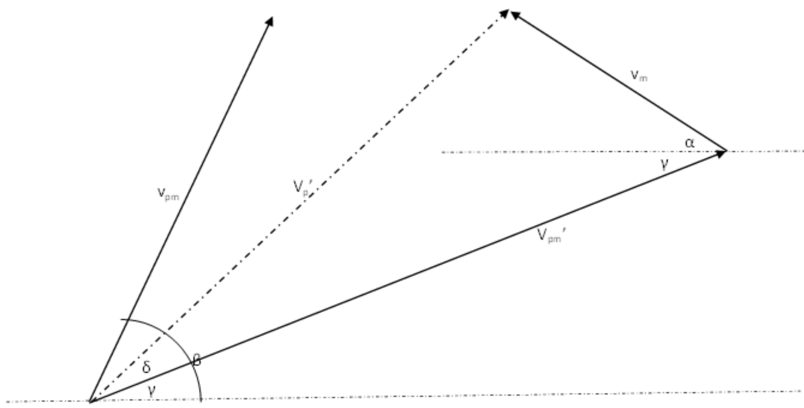

Figure 6: Velocity of the space probe after going around Io

We will set the distance of closest approach to the moon r_m as 1.2 times the radius of the moon to be safe from crashing into it and calculate the necessary impact parameter b based on that. From equation (5) of section 4.1,

$$b = \sqrt{r^2 + \frac{2\mu r_m}{v_{pm}^2}}$$

and from equation (7) of section 5, we know that the speed of the probe relative to the moon is the same, but the direction changes by $180° - 2\theta$, where

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{\frac{\mu^2}{b^2 v_{pm}^4} + 1}}\right)$$

Given this, we calculate γ, the angle v'_{pm} makes with the horizontal, as

$$\gamma = \beta - (180° - 2\theta)$$

The new velocity v'_{pm} will have the same magnitude as v_{pm}, so v'_p, the velocity of the probe relative to Jupiter, is given by

$$v'_p = \sqrt{v_{pm}^2 + v_m^2 - 2v_{pm}v_m\cos(\alpha+\gamma)}$$

$$\frac{v_m}{\sin\delta} = \frac{v'_p}{\sin(\alpha+\gamma)}$$

$$\delta = \sin^{-1}\left(\frac{v_m}{v'_p}\sin(\alpha+\gamma)\right)$$

For simplicity, we want to be able to translate the problem after the probe leaves Io's gravitational field to the problem solved in section 4. This means that we will need the correct impact parameter. Although this may not result in the lowest speed after leaving Io's gravitational field, it does mean that we will not have to waste energy on correcting the path to the circular orbit and changing direction.

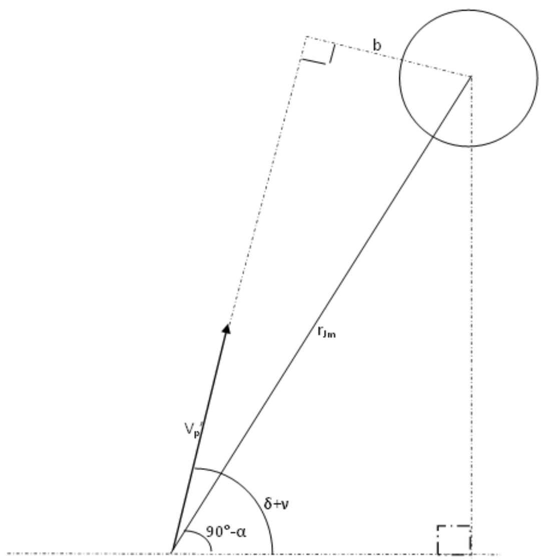

Figure 7: The probe's velocity relative to Jupiter

The probe will go close to the moon relative to the distance to Jupiter, so we can essentially consider it to leave from the moon at the angle of v_p'. Thus, the impact parameter can be calculated as

$$b_1 = r_{Jm} \sin((\gamma + \delta) - (90° - \alpha))$$
$$b_1 = r_{Jm} \sin(\alpha + \gamma + \delta - 90°)$$

and the needed impact parameter is

$$b_2 = \sqrt{r^2 + \frac{2\mu_J r}{(v_p')^2}}$$

from equation (5) of section 4.

6.3 Solving the Equations

The parameter in all of these equations that we can change is α, since we can time the launch of the probe so that the moon will be at the optimal position.

Note that once we set an α value, all of the other values in the equations of the previous section will be determined, and all that's left is to compare if the impact parameter that will come out is the same as the one that is needed. We use the appropriate values for each of the other variables: $v_0 = \sqrt{v_p^2 + v_J^2} = 23.89\,\text{km s}^{-1}$ [6], $v_m = 17.33\,\text{km s}^{-1}$ [7], $r_m = 1.2(\text{radius of Io}) = 2186\,\text{km}$ [7], $\mu = GM_{Io} = 5.958 \times 10^{12}\,\text{N m}^2\ \text{kg}^{-1}$ [7], $r_{Jm} = 1.022 \times 10^6\,\text{km}$ [7], $\mu_J = 1.266 \times 10^{17}\,\text{N m}^2\ \text{kg}^{-1}$ [6], and $r = 6.991 \times 10^4\text{km}$ [6] Using a computer program [8] to test all possible α values between $0°$ and $90°(\frac{\pi}{2}$ radians) in increments of 0.0001 radians, computing b_1 and b_2, and testing when then are sufficiently close, we get that

$$\begin{aligned} \alpha &= 0.8366\,\text{rad} \\ v_p' &= 11.15\,\text{km s}^{-1} \\ \frac{m}{m_0} &= 3.2\% \end{aligned}$$

7 Concluding Remarks

In summary, the results of the previous sections indicate that using the gravity assist maneuver will produce significant fuel savings for space probes. In order to enter Jupiter's orbit at an orbital radius of 1.2 time the radius of Jupiter from an initial speed of 20 km/s, the probe would have to consume 98.5% of its mass as fuel without the use of the gravity assist maneuver, but only 96.8% of its mass with the use of the gravity assist maneuver. This means that the probe can now carry up to twice as much non-fuel materials during its space exploration task. Thus the gravity assist maneuver can greatly increase the efficiency of effectiveness of fuel consumption for future space exploration missions. Although our results are comprehensive under the assumptions stated earlier, further research can still be performed. We can take into consideration the effects of the gravitational fields of other astronomical objects on the space probe while it is performing the gravity assist maneuver. This would increase the accuracy of the amount of fuel consumed. Further research can also include analysis of performing the gravity assist maneuver multiple times on the same or even different moons in order to conserve even more fuel. Overall, the results show that the gravity assist maneuver can indeed produce significant fuel savings for a space probe.

A Python Code to Solve for the Optimal Position of the Moon

```
#Python 2.6
from math import *
mu=5.957577*10**12
rm=1.2*1821300
vs=(20000.0**2+13070**2)**0.5
vm=17000.0
a=0.0001
rjm=1021700000.0
while a<pi/2:
    vsm=(vs**2+vm**2-2*vs*vm*sin(a))**0.5
    beta=acos(vm*cos(a)/vsm)
    b=(rm**2+2*6.67*8.9319*10**11*rm/vsm)**0.5
    theta=asin(1.0/(mu**2/(b**2*vsm**4)+1)**0.5)
    gamma=beta-(180-2*theta)
    vss=(vsm**2+vm**2-2*vsm*vm*cos(a+gamma))**0.5
    delta=asin(vm*sin(a+gamma)/vss)

    b1=((1.2*69911000.0)**2+2*6.67*10**-11*1.8986*10**27*69911000.0*1.2/vss**2)**0.5
    b2=rjm*sin(a+gamma+delta-pi/2)
    if abs(b1-b2)<0.001*b1:
        print a,vss
    a+=0.0001
```

References

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