## Problem B

# 'Foxy Golf Balls' 

TEAM 108


#### Abstract

In this solution, we investigate the feasible club/ball interactions that eventually lead the golf ball to go around the tree (cylinder) and land in the given circular region. The presence of the Magnus force is concluded to be the main reason that drives the ball to fly in a circuitous trajectory around the tree.

To clarify all the interactions within the process, the motion of the ball is divided into two stages - the Free-flight Stage and the Bouncing Stage. For the Free-flight Stage, four parameters (the loft angle, the drift angle, the azimuthal angle and the initial speed) are introduced to determine the initial interaction between the club and the ball. For the Bouncing Stage, we introduce the restitution coefficient and the frictional coefficient to describe the interaction between the ball and the ground.

The dynamic model is established using the Newtonian approach. On a basic level, the trajectory of the ball is numerically solved with a particular set of parameters. To further specify the criteria that lead to satisfying shots, we study the feasible regions in parametric spaces-those regions that incorporate all parameters which result in successful shots-to present a detailed instruction on the golfers' controls. Therefore, the task presented by the problem is accomplished on a technical level.

However, given the fact that absolute accuracies never exist (especially in human controls), we evaluate the margin values of the system. The results reveals that, although some sets of the parameters are viable, the margin values of them are so small that allow for very little deviations in order to lead to the ultimate successful shots. Subsequently, an optimization on margin values is presented, hoping to enhance golfers' chances of success.

Finally, some preliminary results in the presence of wind force are also studies as an extended model.


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## 1 Introduction

Nowadays, quantitative physical analysis is becoming an effective way of studying sports games such as golf, which in turn helps people enhance their performances in those games[2, 3]. In problem B, we are required to find desirable ball/club interactions that will result in a golf ball to go around a tree (modeled as a cylinder with infinite height) and eventually end up in a circular region.

In our solution, the motion of the ball is divided into two stages: (a) the Free-flight Stage, which is initiated by the ball/club interaction, and (b) the Bouncing Stage, which is associated with the terminal interaction between the golf ball and the ground. For both stages, relevant parameters are introduced to describe the interactions.

To present detailed instructions on how to play the desired shots, dynamic model is established with Newtonian Mechanics and solved numerically. Feasible regions in parametric space are calculated, in order to demonstrate the required parameters (such as the initial velocity, the loft angle, the drift angle, the azimuthal angle, etc.) that can lead to successful shots. Moreover, margin values of the system are defined, and used to optimize golfers' chances of success in the presence of the deviations of those parameters. Finally, the influence of wind on the golf game is also studied.

## 2 Definitions and Variables

## Table 1: Notation

| Symbol | Description |
| :---: | :---: |
| M | The mass of golf clubhead. |
| $m$ | The mass of golf ball. |
| $g$ | acceleration of gravity. |
| $R$ | The radius of golf ball. |
| I | The rotational inertia of golf call. |
| $\theta$ | loft angle. |
| $\psi$ | drift angle. |
| $\gamma$ | azimuth angle. |
| $V$ | The velocity of the end point of club shaft. |
| $V_{\text {club }} / V_{\text {club }}^{\prime}$ | The velocity of the end point of club shaft projected in the XOZ/XOY plane. |
| $V_{\text {clubn }} / V_{\text {clubn }}^{\prime}$ | the normal component of $V_{\text {club }} / V_{\text {club }}^{\prime}$. |
| $V_{\text {clubp }} / V_{\text {clubp }}^{\prime}$ | the tangential component of $V_{\text {club }} / V_{\text {club }}^{\prime}$. |
| $V_{c f n} / V_{c f n}^{\prime}$ | the normal component of the club head after hitting in the X OZ/XOY plane. |
| $V_{c f p} / V_{c f p}^{\prime}$ | the normal component of the club head after hitting in the X OZ/XOY plane. |
| $V_{b f p} / V_{b f p}^{\prime}$ | the normal component of the golf ball after hitting in the X OZ/XOY plane. |

## Table 1: (continued)

| $V_{b f p} / V_{b f p}^{\prime}$ | the normal component of the golf ball after hitting in the XOZ/XOY plane. |
| :---: | :---: |
| $\omega_{b f} / \omega_{b f}^{\prime}$ | The angular velocity of golf ball. |
| $C_{D}$ | Drag Coeffcient. |
| $C_{A}$ | constant related to Magnus force. |
| $S$ | effective area. |
| $\rho_{\text {air }}$ | air density. |
| $v_{\\|}$ | horizontal component of landing velocity. |
| $v_{\perp}$ | vertical component of landing velocity. |
| $v_{\\|}^{\prime}$ | horizontal component of the first rebound velocity. |
| $v_{\perp}^{\prime}$ | vertical component of the first rebound velocity. |
| $e$ | restitution Coeffcient. |
| $\mu$ | friction Coeffcient. |
| $L_{b}$ | bounce length. |
| $L_{r}$ | roll distance. |
| $L$ | the total run of the golf ball. |
| $\omega_{b} / \omega_{s}$ | backspin/sidespin of the golf ball. |

## 3 Interactions between Golf Ball and Club

### 3.1 Assumptions

As we know, the interaction between a golf ball and club is an extremely violent collision. The actual contact time is less than 0.5 ms , during which the ball is accelerated from rest to the the speed of over $200 \mathrm{~km} / \mathrm{h}$ [6]. Forces exerted on the golf ball are so large (with values exceeding $5 k N$ ) that solid ball is deformed severely as can be seen in the flash photographs shown by Cochran and Stobbs [7]. In order to simplify the problem, we make the assumptions as follows:

- The collision between a golf club and a golf ball is a completely inelastic collision.
- The club and the ball remains still relative to each other during the interaction.
- The velocity of the end point of club shaft $\vec{V}$ is parallel to the ground.


### 3.2 Theoretical Analysis

According to the experiments done by Cochran and Stobbs, it can be demonstrated that the effect of the club shaft during the collision is not so significant and is negligible compared to the impact force [7]. Assume without loss of generality that $x^{\prime}$-axis is along the velocity of the end point of club shaft (see Fig. 1(a)), thus $\vec{V}=V \overrightarrow{e_{x^{\prime}}}$. The loft angle $\theta$, which stands for the angle between the club shaft and the club head, enables players to launch the golf ball. Also, players can rotate the shaft along its central line to make a drift angle, defined as $\psi$, from the
$x^{\prime}$-axis (see Fig. 1(b)). Therefore, the initial velocity along the $x^{\prime}$-axis and the angular velocity of the ball are decided by the $V, \theta$ and $\psi$.


Figure 1 (a) The configuration of the Golf Course. Two coordinate systems are used in this solution. The $x y z$ frame is fixed on the Golf Course, whereas the $x^{\prime}$-axis of the $x^{\prime} y^{\prime} z^{\prime}$ frame coincides with the velocity of the end point of the club shaft. (b) Illustration of the loft angle $\theta$ and the drift angle $\psi$.

To further clarify the process, we project the surface of the club head and $\vec{V}$ onto the $x^{\prime} o z^{\prime}$ plane and $x^{\prime} o y^{\prime}$ planes (see Fig. 2). In each plane, the oblique impact can be divided into two kinds interactions: the direct impact (which is normal to the surface of the head) and the rolling between the head and the ball (which is tangential to the head surface). In $x^{\prime} o z^{\prime}$ plane, the normal and tangential components of $v_{\text {club }}$ are denoted as $V_{\text {clubn }}$ and $V_{\text {clubp }}$ respectively. We assuming that the velocity components of the ball are $V_{b f n}$ and $V_{b f p}$ when it leaves the club head, and the value of the velocities of the club head change to $V_{c f n}$ and $V_{c f p}$ (see Fig. 2). According to the conservation of linear momentum in both normal and tangential directions, we get

$$
\left\{\begin{align*}
M V_{c f n}+m V_{b f n} & =M V_{c l u b n}  \tag{1}\\
M V_{c f p}+m V_{b f p} & =M V_{c l u b p}
\end{align*}\right.
$$

where $M$ and $m$ are the masses of the club head and the golf ball, respectively. Similarly, the conservation of angular momentum yields

$$
\begin{equation*}
m R V_{b f p}+I \omega_{b f}=0 \tag{2}
\end{equation*}
$$

where $R$ and $I$ are the radius and the rotational inertia of the golf ball, respectively. Due to our first assumption, club head and the ball share the same speed after collision

$$
\left\{\begin{align*}
V_{c f n}-V_{b f n} & =0  \tag{3}\\
V_{b f p}-V_{c f p}-R \omega_{b f}^{\prime} & =0
\end{align*}\right.
$$

Likewise, in the $x^{\prime} o y^{\prime}$ plane we have

$$
\left\{\begin{align*}
M V_{c f n}^{\prime}+m V_{b f n}^{\prime} & =M V_{c l u b n}^{\prime}  \tag{4}\\
M V_{c f p}^{\prime}+m V_{b f p}^{\prime} & =M V_{c l u b p}^{\prime} \\
m R V_{b f p}^{\prime}+I \omega_{b f}^{\prime} & =0 \\
V_{c f n}^{\prime}-V_{b f n}^{\prime} & =0 \\
V_{b f p}^{\prime}-V_{c f p}^{\prime}-R \omega_{b f}^{\prime} & =0
\end{align*}\right.
$$



Figure 2 The velocity and angular velocity components of the club head and the golf ball in the $x^{\prime} o y^{\prime}$ and $x^{\prime} O z^{\prime}$ planes before and after the collision. Two components of $\vec{V}$ are defined as $\mathbf{V}_{\text {club }}$ (in the $x^{\prime} o z^{\prime}$ plane) and $\mathbf{V}_{\text {club }}^{\prime}$ (in the $x^{\prime} o y^{\prime}$ plane), respectively.

To solve Eq. 1 to Eq. 4, we need additional constrains on the $x^{\prime}$ components of all the velocities:

$$
\left\{\begin{array}{l}
V_{c f n} \cos _{\theta}+V_{c f p} \sin \theta-V_{c f n}^{\prime} \cos _{\psi}-V_{c f p}^{\prime} \sin \psi=0  \tag{5}\\
V_{b f n} \cos _{\theta}+V_{b f p} \sin \theta-V_{b f n}^{\prime} \cos _{\psi}-V_{b f p}^{\prime} \sin \psi=0
\end{array}\right.
$$

To present Eq. 1 to Eq. 5 in a more concise version, we use the following two column vectors to contain all the variables respectively, and a coefficient matrix, denoted as $P$, to describe the ten equation.

$$
\left(\begin{array}{c}
M V_{c l u b x}  \tag{6}\\
0 \\
M V_{c l u b z} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right)=\mathbf{P}\left(\begin{array}{c}
V_{c f n} \\
V_{c f p} \\
V_{b f n} \\
V_{b f p} \\
\omega_{b f} \\
V_{c f n}^{\prime} \\
V_{c f p}^{\prime} \\
V_{b f n}^{\prime} \\
V_{b f p}^{\prime} \\
\omega_{b f}^{\prime}
\end{array}\right)
$$

where the matrix P is provided in the appendix A . Therefore, the ball/club interaction has been modeled and clarified till now. By solving Eq. 21, we are able to determine the initial motion of the golf ball. The subsequent free-flight motion will be studied in the following section.

## 4 Free-flight stage

### 4.1 Theoretical Analysis

Once upon the golf ball leaves the surface of the club, its initial conditions, including the velocity and the angular velocity, are determined. The subsequent motion of the ball will be studied with the Newtonian approach. We assume that there is no wind in the Golf Course, thus the golf ball will be affected by gravity, air resistance and Magnus Force.

(a)

(b)

Figure 3 The trajectories of the golf-ball during the Free-flight Stage with parameters of (a) $\theta=\pi / 8, \psi=\pi / 8, \gamma=-\pi / 4$ and $V=58 \mathrm{~m} / \mathrm{s}$; and (b) $\theta=\pi / 6, \psi=\pi / 8, \gamma=-\pi / 4$ and $V=45 \mathrm{~m} / \mathrm{s}$

We denote $f_{\text {air }}$ as the air resistance on the ball. It has been demonstrated [2] that the resistance on the golf ball is in a quadratic form as follows

$$
\begin{equation*}
f_{a i r}=\frac{1}{2} C_{D} \rho_{a i r} S \boldsymbol{v}^{2} \tag{7}
\end{equation*}
$$

where $C_{D}$ represents the drag coefficient, $\rho_{\text {air }}$ represents the air density and $S=\pi R_{\text {ball }}^{2}$ represents aerodynamic cross-section. The drag coefficient depends on the Reynolds number of the system, and the drag coefficient $C_{D}$ depends on the surface condition of the golf ball.

Other than air resistance $f_{\text {air }}$, the golf ball is impacted by another aerodynamic force, namely, the Magnus force $\boldsymbol{f}_{\text {Magnus }}$. The Magnus force is caused by the pressure differences around an object due to its rotation, and is the main reason here that makes the golf ball to go around the tree and, hopefully, hit the circular green. Theoretically, it is in the form of

$$
\begin{equation*}
\boldsymbol{f}_{\text {Magnus }}=C_{M}(\boldsymbol{w} \times \boldsymbol{v}) \tag{8}
\end{equation*}
$$

where $C_{M}$ is identified as the Magnus coefficient and can be expressed as $C_{M}=\frac{\pi^{2}}{2} \rho_{a i r} C_{A} R_{\text {ball }}^{3}$. Here $C_{A}$ represents a constant determined by the features of the golf ball's surface and the internal friction and viscosity of the air (usually satisfies $C_{A} \leq 1$ ).

Qualitatively, the rotations of golf ball can be subdivided into backspin (spin along the y-axis) and sidespin (spin along the z-axis), in which Magnus Effect shows different effects. Backspin placed on a golf ball will allow the ball to gain lift force, thus it will be able to have a much longer flight. Meanwhile, sidespin placed on the ball will allow it to gain side force, which will pull the ball back to the green when it flies out.

On making use of the Newton's second law, and by combining with 7 and Eq.8, we can obtain the dynamic equations during the Free-flight Stage,

$$
\left\{\begin{array}{l}
m \ddot{x}=-\frac{1}{2} C_{D} \rho_{a i r} S v v_{x}+C_{M}\left(w_{y} v z-w_{z} v_{y}\right)  \tag{9}\\
m \ddot{y}=-\frac{1}{2} C_{D} \rho_{a i r} S v v_{y}+C_{M}\left(w_{z} v x-w_{x} v_{z}\right) \\
m \ddot{z}=-\frac{1}{2} C_{D} \rho_{a i r} S v v_{z}+C_{M}\left(w_{x} v y-w_{y} v_{x}\right)-m g
\end{array}\right.
$$

By solving the above ordinary differential equations numerically, we are able to figure out the trajectories, some examples of which are illustrated in Fig. 3.

### 4.2 Differences between Smooth Ball and Dimpled Ball



Figure 4 Illustrations of air flows around (a) smooth ball and (b)dimpled ball.

As mentioned above, the differences in the surface conditions of golf balls (such as smooth ball and dimpled ball) may influence the aerodynamic properties of it. After reviewing literatures, we want to briefly summarize how these surface conditions may effect the motion a golf ball. Most of these results which are based on the bibliography [1] are not original, but they are crucial for rationalizing the dynamic model we established above.

It is demonstrated [1] that the air flow around smooth ball and dimpled ball are shown in figure 4(a)(b). Since the dimpled ball has many dimples on its surface, they will produce some small air vortices. These air vortices will drag the air molecules to the surface of the ball, thus such turbulent flow allows for a smaller separation of air flow around the ball, creating a smaller
area of low pressure behind the ball and causing deceleration at a slower rate than the smooth ball. Therefore, the magnitude of the Magnus force declines a lot with the dimples

The affect of the differences between a smooth ball and a dimpled ball can also be reflected by their drag coefficient $C_{D}$. According to previous researches, drag coefficient changes with Reynold Number Re, and a dimpled ball has a much smaller drag coefficient than a smooth ball[1]. Once upon the drag coefficient declines, the air resistance will consequently decline as well, which means the velocity will decline much smaller. Consequently, the dimpled ball will have more flight time and fly a much longer distances.


Figure 5 Flight distance of a smooth golf ball and a dimpled golf ball. Initial conditions are $V_{x}=40 \mathrm{~m} / \mathrm{s}, V_{y}=10 \mathrm{~m} / \mathrm{s}$

Based on the data in previous works [1], we are able to compared the trajectories of a smooth ball and a dimpled ball by controlling all the parameters the same except their surface conditions. It is illustrated in Fig. 5 that a dimpled ball flies for a further distance than that of a smooth ball.

## 5 Bouncing stage

### 5.1 Assumptions

After hitting the ground, the golf ball will bounce for several times and then roll for a certain distance. The total distance that the ball moves in this stage is largely dependent on landing velocity, spin and characteristic of the ground. The following assumptions are made to simplify flight model as follows:

- Major part of the mechanical energy is dissipated during the first bounce.
- The frictional force is so great that the velocity of contact point falls to zero when leaving the green, i.e. a state of pure rolling after the collision with the ground.
- The sidespin will not effect the motion parallel to the ground.
- The coefficient of restitution and the coefficient of friction that we cited from literatures are reasonable.


### 5.2 Theoretical Analysis



Figure 6 (a) The angular velocity and the velocity components of the golf ball before and after the bounce. The two components of landing velocity are defined as $v_{\|}$and $v_{\perp}$, and the two components of rebound velocity are defined as $v_{\|}^{\prime}$ and $v_{\perp}^{\prime}$. (b) Illustration of the trajectory of bouncing and rolling when $v_{\|}^{\prime}$ is negative.

The landing velocity is decomposed into the horizontal component $v_{\|}$and vertical component $v_{\perp}$ in order to divide the oblique impact into two kinds of interactions (the direct impact normal to the surface, and the interaction tangential to the surface). Define $v_{\|}^{\prime}$ and $v_{\perp}^{\prime}$ as the velocity after the first rebound, and $L_{b}$ and $L_{r}$ as bounce length and roll distance respectively. Since the frictional torque does not do any work to the system, the conservation of angular momentum before and after each single bounds yields

$$
\begin{equation*}
m R v_{\|}-I \omega_{r}=m R v_{\|}^{\prime}+I \omega_{r}^{\prime} \tag{10}
\end{equation*}
$$

where $\omega_{r}^{\prime}=-v_{\|}^{\prime} / R$. Therefore, the rebound velocities and the angular velocity are given by

$$
\left\{\begin{align*}
v_{\perp}^{\prime} & =e\left|v_{\perp}\right|  \tag{11}\\
v_{\|}^{\prime} & =\left(\frac{5}{7}\right) v_{\|}-\left(\frac{2}{7}\right) R \omega_{r}
\end{align*}\right.
$$

We can see that $\omega_{r}$ contributes to the the change of $v_{\|}$after every bounce. But the energy is largely dissipated during the first bounce and $\omega_{r}^{\prime}$ becomes small. It is assumed that the horizontal velocity is constant after the first bounce, i.e. $v_{\|}^{\prime}$.

The coefficient of restitution $e$ between a golf ball and the ground has been measured by Penner[6]. It was found that the value of $e$ decreased with increasing impact speed with the following function providing a good fit to the data (see in appendix C):

$$
\begin{cases}e=0.510-0.0375\left|v_{\perp}\right|+0.000903\left|v_{\perp}\right|^{2} & \text { when }\left|v_{\perp}\right| \leq 20 \mathrm{~m} / \mathrm{s}  \tag{12}\\ e=0.120 & \text { when }\left|v_{\perp}\right|>20 \mathrm{~m} / \mathrm{s}\end{cases}
$$

It has been argued that the coefficient of restitution with larger impact speed is smaller. Hence the coefficient of restitution, which is the ratio of impulse of restitution to impulse of compression, is little. In spite of hitting the green with high speed, the vertical velocity will drastically decrease during the first bounce. From the literature we found, the landing velocity for a typical drive ranges from $26.8 \mathrm{~m} / \mathrm{s}$ to $35 \mathrm{~m} / \mathrm{s}$. Furthermore, the sidespin will contribute to the deformation of the turf. Consequently $e$ becomes smaller. Therefore, the vertical velocity of first rebound can be expressed as

$$
\begin{equation*}
v_{\perp}^{\prime}=0.12\left|v_{\perp}\right| \tag{13}
\end{equation*}
$$

After the first bounce, $v_{\|}^{\prime}$ will be relatively small, in which case $e$ will be approximately equal to 0.5 for the second and subsequent bounces on green.

The only difference between discussed case and Daish's case is that the golf ball have not only backspin $\omega_{r}$ but also sidespin $\omega_{s}$. In the process of collision, the sidespin of the ball fall to zero due to the great friction and it will not contribute to the change of $v_{\|}$. Hence,the model of the run is practical[6]. Ignoring the air drag, it is a oblique projectile motion with initial velocity $v_{\|}^{\prime}$ and $v_{\perp}^{\prime}$, thus the first bounce length will be given by:

$$
\begin{equation*}
L_{b 1}=\frac{2 v_{\|}^{\prime} v_{\perp}^{\prime}}{g} \tag{14}
\end{equation*}
$$

And the subsequent bounce length is a geometric progression with common ratio $e(e=0.5)$. Therefore,the total bounce length is:

$$
\begin{equation*}
L_{b}=\left(\sum_{i=0}^{n} e^{i}\right) L_{b 1}=2 L_{b 1} \tag{15}
\end{equation*}
$$

After the bound phase, the golf ball will be rolling along the green with a initial speed of $v_{\perp}^{\prime}$ and the acceleration $-\mu g$, where $\mu$ is the coefficient of friction. The roll distance will be given by

$$
\begin{equation*}
L_{r}=\frac{v_{\|}^{\prime 2}}{2 \mu g} \tag{16}
\end{equation*}
$$

$\mu$ is set to be 1.0 , which has a good agreement with the experiment. Finally, the run of the golf will be given by $L=L_{b}+L_{r}$.

In the problem, the backspin $\omega_{r}$ is too large that $v_{\|}^{\prime}$ can be negative, which can be seen in the formula. Fig. 6(b) shows the effect backspin has on the run of the a golf ball when $v_{\|}^{\prime}$ is negative. When simulating the problem,let

$$
\left\{\begin{align*}
v_{z} & =v_{\|}  \tag{17}\\
v_{x}^{2}+v_{y}^{2} & =v_{\|}^{2} \\
\omega_{x}^{2}+\omega_{y}^{2} & =\omega_{s}^{2}
\end{align*}\right.
$$

As discussed above, $\omega_{x}$ will effect the $v_{y}^{\prime}$ and the $\omega_{y}$ will effect the $v_{x}^{\prime}$. then

$$
\left\{\begin{array}{r}
v_{z}^{\prime}=0.12 v_{z}  \tag{18}\\
v_{x}^{\prime}=\left(\frac{5}{7}\right) v_{x}-\left(\frac{2}{7}\right) R\left|\omega_{y}\right| \\
v_{y}^{\prime}=\left(\frac{5}{7}\right) v_{y}-\left(\frac{2}{7}\right) R\left|\omega_{x}\right|
\end{array}\right.
$$

Therefore, the bounce lengths, or alternatively, the displacements after the Bouncing Stage, are

$$
\left\{\begin{array}{l}
\Delta x=\frac{4 v_{z}^{\prime} v_{x}^{\prime}}{g}+\frac{v_{x}^{\prime 2}}{2 \mu g}  \tag{19}\\
\Delta y=\frac{4 v_{z}^{\prime} v_{y}^{\prime}}{g}+\frac{v_{y}^{\prime 2}}{2 \mu g}
\end{array}\right.
$$

## 6 Results

### 6.1 Parametric space and feasible regions

A feasible trajectory must go around the tree (cylinder) and land in the given circular region. By calculating different trajectories with all possible initial conditions, all feasible parameters could be obtained. In Fig. 7(b), we calculate the feasible region with initial conditions $-\pi / 2<$ $\gamma<0,0<\psi<\pi / 2, V_{\text {club }}=58 \mathrm{~m} / \mathrm{s}, \theta=\pi / 8$.


Figure 7 All feasible trajectory. (Red line represents the free flight stage. Blue line represents the bouncing stage.Light green region represents tree.Dark green region represents green.)

Still and all, a golfer cannot get enough information to make a successful shoot using results in the sections above. He cares more about proper angel and power that results in a feasible trajectory.

We assume that a well-trained golfer can control the power used in every shoot very precisely, and that $V_{\text {clud }}$ does not change in our problem. Once a specific type of club is chose, the loft angel is then fixed (which means $\theta$ is constant). We consider that, golfer can control the drift angel $\theta$ and azimuth angel $\gamma$ to shoot a proper trajectories. By scanning the parametric space, the Fig. 7 demonstrates the feasible regions have central symmetry in parametric space. By choosing parameters in these regions, the golfer will ultimately lead to a successful shoot.

### 6.2 Margin values of the system



Figure 8 Feasible regions in parametric space. (The red region represents desirable parameters.)

One cannot deny the fact that, no matter how well-trained a golfer is, errors do occur when his/her controls a club and make a shot. Fig. 8 shows the feasible region is quite narrow in the parametric space. Under these circumstances, the golfers have to make their controls so precise - the precision that sometimes even impossible achieve - to lead to a desirable shot. Therefore, we want to the make problem easier for golfer to shoot successfully. To measure the 'difficulty' golfers have to face with, we define the margin value of the system (which is the
root-mean square of the feasible region)

$$
\begin{equation*}
M=\frac{\sqrt{\int_{\text {FeasibleRegion }}\left((\psi-\bar{\psi})^{2}+(\gamma-\bar{\gamma})^{2}\right) d \tau}}{S} \tag{20}
\end{equation*}
$$

Qualitatively speaking, the larger the margin value becomes, the large the feasible regions would be. Consequently, albeit some deviations might exist in golfers' controls, they are still able to accomplish the shot and lead the ball to go around the tree. Fig. 9 shows that, the margin value under condition of $50<V_{\text {club }}<70$ is much larger than that under other conditions. As a conclusion, we could determine the most suitable loft angel is around $\pi / 8$.


Figure 9 The margin value of the system. It can be observed that the margin value reaches the maximum when the velocity is around $50 \mathrm{~m} / \mathrm{s} \sim 70 \mathrm{~m} / \mathrm{s}$, within which the golfers' chances to successfully accomplish the task is enhanced.

### 6.3 Extended model

The real condition is much more complex than the parameters considered in theory, and the most important one is wind. It's easy to imagine that, with a side wind making a successful shoot becomes easier. Given a wind (speed: $5 \mathrm{~m} / \mathrm{s}$ along y axes) Fig. 10(a) demonstrates a fact that one of the feasible region become much larger than the same region in Fig. 10(b), in which condition parameters are the same except wind speed.

Fig. 10 shows obviously what influence is caused by wind.

## 7 Conclusion

As discussed in our article, Magnus force is significant to solve the problem. In the freeflight bouncing, the Magnus force plays two important roles, one of which caused by backspin provide the lift force. Another one caused by sidespin drives the ball to fly in a circuitous trajectory around the tree.


Figure 10 Feasible regions with wind.

After studying the correlation between feasible trajectories and the way hitting the short, we can instruct the golfer in the problem how to hit a successful shot. The golfer is suggested to use the club with the loft angle being $22.5^{\circ}$, rotate the shaft with the drift angle being around $17^{\circ}$ and swing in the direction $44^{\circ}$ away from the tree and at the speed of $60 \pm \mathrm{m} / \mathrm{s}$.

## 8 Strengths and Weaknesses

### 8.1 Strengths

- We took all interactions into account including that between ball and club, ball and ground, ball and air. The result is accurate and convincing.
- Knowing the kind of club, drift angel, azimuthal angel and impact power we could tell whether the trajectory is proper.
- Using the $\psi-\gamma$ diagram we gave instructive suggestion about how to make a successful shoot. By investigating 'Margin Value', the difficulty of hitting target can be described qualitatively.


### 8.2 Weaknesses

Our weaknesses are mainly caused by the rough models we use, especially the shortcomings and imperfections of our assumptions, which are a little different from the reality.

- We assume the collision between golf ball and golf club is a completely inelastic collision. Actually, the real situation is imperfect inelastic collision, which will give the ball more initial velocity during the hit.
- In the bouncing stage, we assume that the ground is flat. The pressure act on the golf ball is vertical and the friction is horizontal. In fact, the golf ball will deform the green surface so that the surface of the ground will not be flat during the collision. This influent on the bouncing stage can be further discussed.


## References

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## Appendix A Matrix Form

Matrix Form:

$$
\left(\begin{array}{c}
M V_{c l u b x}  \tag{21}\\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right)=\mathbf{P}\left(\begin{array}{c}
V_{c f n} \\
V_{c f p} \\
V_{b f n} \\
V_{b f p} \\
\omega_{b f} \\
V_{c f n}^{\prime} \\
V_{c f p}^{\prime} \\
V_{b f n}^{\prime} \\
V_{b f p}^{\prime} \\
\omega_{b f}^{\prime}
\end{array}\right)
$$

$P_{x}:$
$P_{y}:$
$P_{z}:$
$\omega_{y}:$
$\omega_{z}:$$\quad\left(\begin{array}{cccccccccc}M \cos \theta & M \sin \theta & m \cos \theta & m \sin \theta & 0 & M \cos \psi & M \sin \psi & m \cos \psi & m \sin \psi & 0 \\ 0 & 0 & 0 & 0 & 0 & -M \sin \psi & M \cos \psi & -m \sin \psi & m \cos \psi & 0 \\ \text { Collision }: & M \sin \theta & -M \cos \theta & m \sin \theta & -m \cos \theta & 0 & 0 & 0 & 0 & 0 \\ \text { Collision }: & 0 & 0 & 0 & m R & I & 0 & 0 & 0 & 0 \\ \text { Non-slip }: \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m R & I \\ \text { Non-slip }: \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_{x}: & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 \\ 0 & -1 & 0 & 1 & -R & 0 & 0 & 0 & 0 & 0 \\ \cos \theta & \sin \theta & \cos \theta & \sin \theta & 0 & -\cos \psi & -\sin \psi & -\cos \psi & -\sin \psi & 0\end{array}\right)$
(22)

Row (1)-(3) are the conservations of linear momenta along the normal and parallel directions. Row (4)-(5) are the conservations of angular momenta. Row (6)-(9) indicate that after the collision, the loft and the ball share the same speed and direction. The last equation is the velocity constrain in this coordinate.

## Appendix B Mathematica Program

## B. 1 Start Condition.nb

```
% This program mainly generates initial data.
ClearAll["Global`*"];
R=4.27*10^-2/2
m=45.9*10^-3;
M = 0.2;
i}=0.4*45.9*1\mp@subsup{0}{}{\wedge}-3*4.27*1\mp@subsup{0}{}{\wedge}-2*4.27*1\mp@subsup{0}{}{\wedge}-2/
c = 0.25;
\[Rho] = 1.29;
A = \[Pi] R^2;
S = 5*10^-5;
g = 9.8;
\[Mu] = 1;
(*
To Calculate the start condition of flying stage using parameters \
```

controled by golfer.
*)
Matrix $=\{\{M \operatorname{Cos}[t h e t a], M \operatorname{Sin}[t h e t a], m \operatorname{Cos}[t h e t a], m \operatorname{Sin}[t h e t a], 0$,
M Cos[psi], M Sin[psi], m Cos[psi], m Sin[psi], 0\},
\{0, 0, 0, 0, 0, -M Sin[psi], M Cos[psi], -m Sin[psi], m Cos[psi],
$0\}$,
$\{M \operatorname{Sin}[t h e t a],-M \operatorname{Cos}[t h e t a], m \operatorname{Sin}[t h e t a],-m \operatorname{Cos}[t h e t a], 0,0,0$,
$0,0,0\}$,
$\{0,0,0, m R, i, 0,0,0,0,0\}$,
$\{0,0,0,0,0,0,0,0, m \mathrm{R}, \mathrm{i}\}$
$\{1,0,-1,0,0,0,0,0,0,0\}$,
$\{0,0,0,0,0,1,0,-1,0,0\}$,
$\{0,-1,0,1,-R, 0,0,0,0,0\}$,
$\{0,0,0,0,0,0,-1,0,1,-R\}$,
\{Cos[theta], Sin[theta], Cos[theta], Sin[theta],
0, -Cos[psi], -Sin[psi], -Cos[psi], -Sin[psi], 0\}\};
ans $=$ LinearSolve[Matrix, $\{M \mathrm{~V}, 0,0,0,0,0,0,0,0,0\}]$;

\[Omega]y[theta_, psi_, V_] :=
Evaluate[Simplify[
ans[[5]], (-\[Pi]/2 <

```
    psi) && (psi < \[Pi]/2) && (theta < \[Pi]/
    2) && (theta > -\[Pi]/2)]];
\[Omega]z[theta_, psi_, V_] :=
    Evaluate[-Simplify[
        ans[[10]], (-\[Pi]/2 <
            psi) && (psi < \[Pi]/2) && (theta < \[Pi]/
            2) && (theta > -\[Pi]/2)]];
vx[theta_, psi_, v_] :=
    Evaluate[Simplify[
        ans[[3]]*Cos[theta] + ans[[4]]*Sin[theta] + ans[[8]]*Cos[psi] +
            ans[[9]]*
            Sin[psi], (-\[Pi]/2 <
            psi) && (psi < \[Pi]/2) && (theta < \[Pi]/
                    2) && (theta > -\[Pi]/2)]];
vy[theta_, psi_, V_] :=
    Evaluate[Simplify[-ans[[8]]*Sin[psi] +
        ans[[9]]*
            Cos[psi], (-\[Pi]/2 <
            psi) && (psi < \[Pi]/2) && (theta < \[Pi]/
            2) && (theta > -\[Pi]/2)]];
vz[theta_, psi_, v_] :=
    Evaluate[Simplify[
        ans[[3]]*Sin[theta] -
            ans[[4]]*
            Cos[theta], (-\[Pi]/2 <
            psi) && (psi < \[Pi]/2) && (theta < \[Pi]/
            2) && (theta > -\[Pi]/2)]];
```


## B. 2 Possible Track.nb

```
(*
To Calculate a possible track with start condition.
*)
\[Gamma] = \[Pi]/4
equ = {m x''[t] == -0.5*
    c \[Rho] A Sqrt[(x'[t]^2 + y'[t]^2 + z'[t]^2)] x'[t] +
```

```
        S (\[Omega]y[theta, psi, V] z'[t] - \[Omega]z[theta, psi, V] y'[
        t]), m y''[t] == -0.5*
        c \[Rho] A Sqrt[(x'[t]^2 + y'[t]^2 + z'[t]^2)] y'[t] +
            S (\[Omega]z[theta, psi, V] x'[t]),
    m z''[t] == -0.5*c \[Rho] A Sqrt[(x'[t]^2 + y'[t]^2 + z'[t]^2)]
    z'[t] + S (-\[Omega]y[theta, psi, V] x'[t]) - m g, x[0] == 0,
    y[0] == 0, z[0] == 0, x'[0] == vx[theta, psi, V],
    y'[0] == vy[theta, psi, V], z'[0] == vz[theta, psi, V]};
obs[\mp@subsup{x}{-}{\prime},\mp@subsup{y}{-}{\prime},\mp@subsup{z}{-}{\prime,}\[Gamma]_] :=
    If[((10 Cos[\[Gamma]] - x)^2 + (10 Sin[\[Gamma]] - y)^2 <=
    10) || (z <= 0), True, False];
green[x_, y_, \[Gamma]_] :=
    If[((120 Cos[\[Gamma]] - x)^2 + (-120 Sin[\[Gamma]] - y)^2 <= 100),
    True, False];
temp = Reap[
    NDSolve[equ /. {theta -> \[Pi]/8, psi -> \[Pi]/8, V -> 58}, {x, y,
        z},{t, 0, 80},
    Method -> {"EventLocator",
        "Event" -> (obs[x[t], y[t], z[t], \[Gamma]]),
        "EventAction" :> Sow[{t, x'[t], y'[t], z'[t]}]}]];
trac = temp[[1]];
final = temp[[2]][[1]][[1]][[1]];
vxf = temp[[2]][[1]][[1]][[2]];
vyf = temp[[2]][[1]][[1]][[3]];
vzf = 0.12 temp[[2]][[1]][[1]][[4]];
\[Omega] = \[Omega]y[\[Pi]/6, -\[Pi]/4, 80];
vxfn = 5/7 vxf - 2/7 R \[Omega];
vyfn = 5/7 vyf - 2/7 R \[Omega];
gTrac = ParametricPlot[{Cos[\[Gamma]] x[t] - Sin[\[Gamma]] y[t],
    Sin[\[Gamma]] x[t] + Cos[\[Gamma]] y[t]} /. trac, {t, 0, final},
    AspectRatio -> Automatic, PlotRange -> {{0, 150}, {-45, 45}},
    Frame -> True, PlotStyle -> {Red, Thick},
    LabelStyle -> Directive[Thick, Black],
    AxesLabel -> {Style["X", 15], Style["Y", 15]}];
ox = (x[final] /. trac)[[1]];
oy = (y[final] /. trac)[[1]];
nx = (x[final] /. trac)[[1]] + 4/g vzf vxfn + vxfn^2/(2 \[Mu] g);
ny = (y[final] /. trac)[[1]] + 4/g vzf vyfn + vyfn^2/(2 \[Mu] g);
gRoll = Graphics[{Thick, Blue,
```

```
    Line[{{Cos[\[Gamma]] ox - Sin[\[Gamma]] oy,
    Sin[\[Gamma]] ox + Cos[\[Gamma]] oy}, {Cos[\[Gamma]] nx -
    Sin[\[Gamma]] ny, Sin[\[Gamma]] nx + Cos[\[Gamma]] ny}}]}];
Show[gTrac, gTree, gGreen, gTrac, gRoll]
```


## B. 3 Feasible Region.nb

```
(*
Find feasible region in parametric space
*)
Ap = {}; Ag = {};
Do[
    Do [
    temp = Reap[
        NDSolve[equ /. {theta -> \[Pi]/8, V -> 58}, {x, y, z}, {t, 0,
        10}, Method -> {"EventLocator",
            "Event" -> (obs[x[t], y[t], z[t], \[Gamma]]),
            "EventAction" :> Sow[{t, x'[t], y'[t], z'[t]}]}]];
    trac = temp[[1]];
    (*
    The Same code in last paragraph.
    *)
    If[green[nx, ny, \[Gamma]], AppendTo[Ap, psi];
    AppendTo[Ag, \[Gamma]]], {\[Gamma], -\[Pi]/2, -\[Pi]/8,
    0.02}], {psi, 0, \[Pi]/4, 0.02}]
Show[Graphics[{LightBlue, Rectangle[{0, 0}, {-\[Pi]/2, \[Pi]/2}]},
    Axes -> True,
    AxesLabel -> {Style["\[Gamma]", 15], Style["\[Psi]", 15]},
    AxesStyle -> Directive[Black, 8]],
ListPlot[Transpose[{Ag, Ap}],
    PlotRange -> {{0, -\[Pi]/2}, {0, \[Pi]/2}},
    PlotMarkers -> Automatic, PlotStyle -> Red],
AxesStyle -> Directive[Black, 12],
Ticks -> {{0, -Pi/4, -Pi/2}, {\[Pi]/2, \[Pi]/4, 0}}]
```


## B. 4 Wind influence.nb

```
(*
Calculate the track with the influence of wind.
*)
w = -5;
equW = {m x''[t] == -0.5*
            c \[Rho] A Sqrt[(x'[t]^2 + (y'[t] - w)^2 + z'[t]^2)] x'[t] +
            S (\[Omega]y[theta, psi, V] z'[t] - \[Omega]z[theta, psi,
            V] (y' [t] - w)),
        m y''[t] == -0.5*
            c \[Rho] A Sqrt[(x'[t]^2 + (y'[t] - w)^2 + z'[t]^2)] y'[t] +
            S (\[Omega]z[theta, psi, V] x'[t]),
        m z''[t] == -0.5*
            c \[Rho] A Sqrt[(x'[t]^2 + (y'[t] - w)^2 + z'[t]^2)] z'[t] +
            S (-\[Omega]y[theta, psi, V] x'[t]) - m g, x[0] == 0, y[0] == 0,
            z[0] == 0, x'[0] == vx[theta, psi, V], y'[0] == vy[theta, psi, V],
            z'[0] == vz[theta, psi, v]};
Ap = {}; Ag = {};
Do[
Do [
    temp = Reap[
            NDSolve[equW /. {theta -> \[Pi]/8, V >> 58}, {x, y, z}, {t, 0,
                10}, Method -> {"EventLocator",
                    "Event" -> (obs[x[t], y[t], z[t], \[Gamma]]),
            "EventAction" :> Sow[{t, x'[t], y'[t], z'[t]}]}]];
            (*
    The Same code in last paragraph.
            *)
    If[green[nx, ny, \[Gamma]], AppendTo[Ap, psi];
    AppendTo[Ag, \[Gamma]]], {\[Gamma], -\[Pi]/3, -\[Pi]/10,
    0.01}], {psi, \[Pi]/20, \[Pi]/7, 0.01}]
Show[Graphics[{LightBlue, Rectangle[{0, 0}, {-\[Pi]/2, \[Pi]/2}]},
    Axes -> True,
AxesLabel -> {Style["\[Gamma]", 15], Style["\[Psi]", 15]},
AxesStyle -> Directive[Black, 8]],
ListPlot[Transpose[{Ag, Ap}],
PlotRange -> {{0, -\[Pi]/2}, {0, \[Pi]/2}},
```

PlotMarkers -> Automatic, PlotStyle -> Red],
AxesStyle -> Directive[Black, 12],

Ticks $->\{\{0,-\backslash[P i] / 4,-\backslash[P i] / 2\},\{\backslash[P i] / 2, ~ \[P i] / 4,0\}\}]$

## Appendix C Mechanical Parameters




Figure 11 Related mechanical parameters obtained by experiments [1, 6].

