TEAM#340

Problem A

ION THRUSTERS TO SATURN

Abstract

This paper analysed the voyage of a spacecraft powered by ion thrusters from Earth to Saturn. A Conic Section Splicing Model was established to describe the process. We divided the entire process into five stages, including Earth-Sun Transfer Phase, First Orbit Sun Phase, Venus Swing by Phase, Second Orbit Sun Phase and Sun-Saturn Transfer Phase.

In Earth-Sun Transfer Phase, we established 5 equations to obtain the kinetic differential equation of the spacecraft. Since it is a higher order nonlinear non-homogeneous equation with no analytic solution, we could only derive the numerical solution using MATLAB.

In First and Second Orbit Sun Phase, using law of conservation of angular momentum, law of conservation of mechanical energy and the geometric relationship of orbits, we find the velocities when the spacecraft has just escaped Planet A and has just interacted with Planet B.

In Venus Swing by Phase, instead of studying the gravitational slingshot effect of Venus in dynamics, we chose to easily calculate the velocities of the spacecraft before and after swinging by Venus directly from the geometric relationships of the first and second elliptical orbits around Sun.

In Sun-Saturn Transfer Phase, aim at saving fuel, we firstly let the spacecraft perform free motion under the gravity of Jupiter, then turn on the ion thrusters to slow it down until reaching the prescribed orbit. Use MATLAB to find the trajectory of the spacecraft's motion.

In summary, the minimum amount of fuel it takes is 2402.77kg. The duration of the trip is 4923.89 days (13.49 years). During Stage 1 ,turn on the ion thrusters in the same direction with the angular velocity for 65452.66 hours. While during Stage 5 , turn on the ion thrusters in the opposite direction of the angular velocity for 0.43 hours.

Contents

1 Introduction

On October 15, 1997, Cassini-Huygens, which involved a collaboration between NASA, ESA and ASI, was launched into space to study the planet Saturn and its system. Cassini, the first space probe to enter Saturn's orbit, spent 7 years on the voyage to Staurn, which included flybys of Venus, Earth, the asteroid 2685 Masursky, and Jupiter.

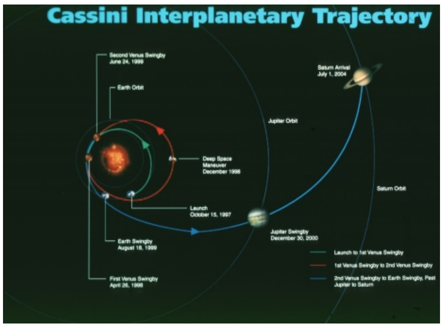


Fig1. Cassini Interplanetary Trajectory

According to the assignment, our work begins at the point when the spacecraft escapes Earth at a specific velocity and ends when it falls into the prescribed circular orbit around Saturn. The spacecraft is only powered by the equipped ion thrusters, which is required to take the minimum amount of fuel. Based on the Cassini's experience and the requirements of this voyage, it is difficult to provide enough energy only by the equipped thrusters, so we need to use the gravitational slingshot effect of the planets in solar system to accelerate the spacecraft. Moreover, since this voyage doesn't impose time restrictions but only considers fuel consumption, we don't have to accelerate the spacecraft to a fast velocity to save time. To summarize the foregoing, we decided to use the gravitational slingshot effect of Venus to accelerate the spacecraft once, and then let it directly fall into the prescribed orbit around Saturn.

To realize the voyage from Earth to Saturn, we established a Conic Section Splicing Model, which was divided into five stages, including Earth-Sun Transfer Phase, First Orbit Sun Phase, Venus Swing by Phase, Second Orbit Sun Phase and Sun-Saturn Transfer Phase. With the Conic Section Splicing Model and discussion, the flight plan taking the minimum amount of fuel can hopefully be found after calculation run on computers.

2 Assumptions and Notations

2.1 Basic Assumptions for Our Models

- Assume that the spacecraft is only exposed to the sun's gravity during the flight when it is free from Planet A bondage and is not captured by Planet B. Because the gravity from both Planet A and Planet B are very small contrasted to the sun's gravitational attraction during this period. And ignoring other gravities can simplify the calculation and facilitate the model establishment.
- Suppose all the planets are in the ideal position. In other words, the spacecraft happens to be captured by planet B when it is at the right velocity. It is theoretically possible to find a right launch time to achieve such an ideal result. In this paper we deal with it briefly.
- Assume the paths of all the planets orbiting the sun are circular. Since the eccentricities of the motions of the planets in solar system are very small, we can simplify the calculation by approximating them to circles.
- Suppose all objects in the system can be considered as mass points.
- Ignore the effects of planet's rotation.
- Ignore the effects of relativity.

2.2 Notations

Variable	Explanations
$\boldsymbol{m}$	The mass of the spacecraft (kg)
$\boldsymbol{M_E}$	The mass of Earth ($M_E = 5.965 \times 10^{24} kg$)
$\boldsymbol{M_V}$	The mass of Venus ($M_V = 4.869 \times 10^{24} kg$)
$\boldsymbol{M_S}$	The mass of Saturn ($M_S = 5.6846 \times 10^{26} kg$)
$\boldsymbol{M_{SUN}}$	The mass of Sun ($M_{SUN} = 1.9891 \times 10^{30} kg$)
$\boldsymbol{R_E}$	The orbital semi-major axis of Earth ($R_E = 1.4960 \times 10^{8} km$)
$\boldsymbol{R_V}$	The orbital semi-major axis of Venus ($R_V = 1.0821 \times 10^{8} km$)
$\boldsymbol{R_S}$	The orbital semi-major axis of Saturn ($R_S = 1.4334 \times 10^{9} km$)
$\boldsymbol{r_E}$	The orbital radius of the spacecraft around Earth (km)

$\boldsymbol{r_S}$	The orbital radius of the spacecraft around Saturn (km)
$\boldsymbol{T_E}$	The orbital period of the spacecraft around Earth ($T_E = 1.5h$)
$\boldsymbol{T_S}$	The orbital period of the spacecraft around Saturn ($T_S = 40h$)
G	Gravitational constant ($\mathrm{G} = 6.6726 \times 10^{-11} N \cdot m^2 \cdot kg^{-1}$)
$\boldsymbol{g_0}$	Standard gravitational acceleration ($g_0 = 9.8067 m \cdot s^{-2}$)
$\boldsymbol{a_1}$	The semi-long axis of the spacecraft's elliptical orbit to Venus by Sun's gravity (km)
$\boldsymbol{b_1}$	The semi-short axis of the spacecraft's elliptical orbit to Venus by Sun's gravity (km)
$\boldsymbol{a_2}$	The semi-long axis of the spacecraft's elliptical orbit to Staurn by Sun's gravity (km)
$\boldsymbol{b_2}$	The semi-short axis of the spacecraft's elliptical orbit to Saturn by Sun's gravity (km)
$\boldsymbol{v_0}$	The initial velocity of the spacecraft orbiting Earth (Under Earth Reference System)($\mathrm{km} \cdot s^{-1}$)
$\boldsymbol{v_1}$	The velocity of the spacecraft escaping Earth (Under Earth Reference System)($\mathrm{km} \cdot s^{-1}$)
$\boldsymbol{v_2}$	The velocity of the spacecraft which happened to interact with Venus (Under Sun Reference System)($\mathrm{km} \cdot s^{-1}$)
$\boldsymbol{v_3}$	The velocity of the spacecraft escaping Venus (Under Sun Reference System)($\mathrm{km} \cdot s^{-1}$)
$\boldsymbol{v_4}$	The velocity of the spacecraft which happened to interact with Saturn (Under Sun Reference System)($\mathrm{km} \cdot s^{-1}$)
$\boldsymbol{v_5}$	The final velocity of the spacecraft orbiting Saturn (Under Saturn Reference System)($\mathrm{km} \cdot s^{-1}$)

3 Physical and Geometric Analysis of Model

To realize the voyage from Earth to Saturn, we established a Conic Section Splicing Model, which was divided into five stages, including Earth-Sun Transfer Phase, First Orbit Sun Phase, Venus Swing by Phase, Second Orbit Sun Phase and Sun-Saturn Transfer Phase.

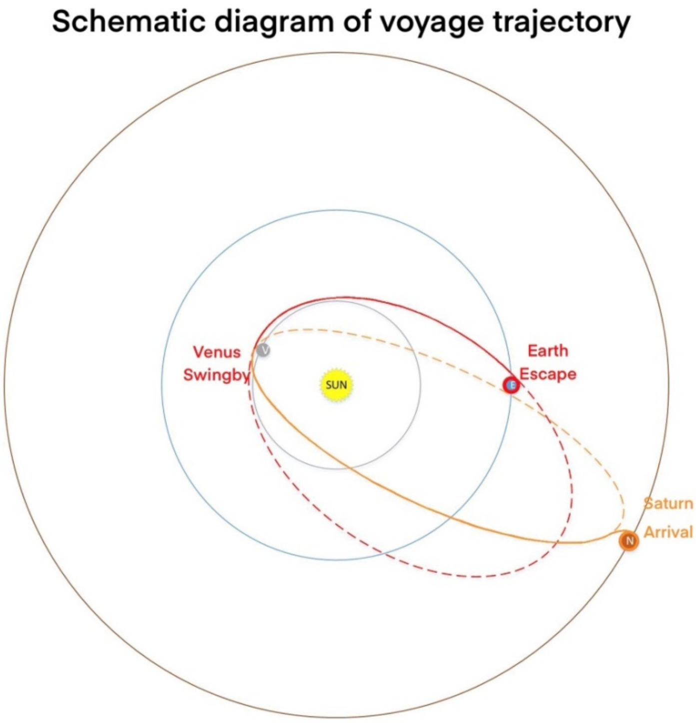


Fig2. Schematic diagram of voyage trajectory

3.1 Earth-Sun Transfer Phase

Under Earth Reference System, when the spacecraft orbits Earth in a circle with an orbital period of 90 minutes, $r_E = 6.6493 \times 10^3 km$, $v_0 = 7.7368 km/s$. Establish a polar coordinate system with the geocentric center as the polar point and the rays from the geocentric center past the initial position of the spacecraft as the polar axis.

When the spacecraft use ion thrusters to accelerate,

$$\vec{F} = \left(0.4, \frac{\pi}{2} + \theta\right) \tag{1.1}$$

$$\vec{G} = \left(\frac{GM_E m}{\rho^2}, -\theta\right) \tag{1.2}$$

$$\mathrm{m} = m_0 - \frac{|\vec{F}|}{g_0 I_{sp}} \tag{1.3}$$

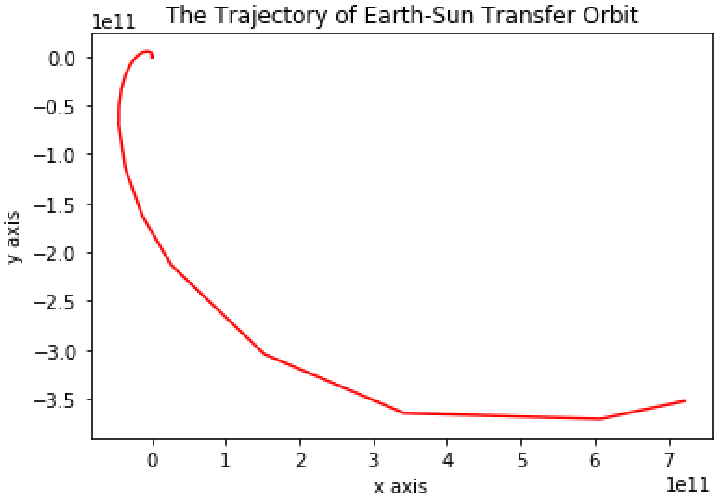
$$a_\rho = \ddot{\rho} - \rho \cdot \dot{\theta}^2 \tag{1.4}$$

$$a_\theta = \rho \cdot \ddot{\theta} - 2\dot{\rho}\dot{\theta} \tag{1.5}$$

By associating these five equations, we can obtain the kinetic differential equation of the spacecraft. Since it is a higher order nonlinear non-homogeneous equation with no analytic solution, we use the Coutarone algorithm included in MATLAB to obtain the numerical solution by interpolating the integral. We can only derive the numerical solution using MATLAB.

After MATLAB's calculation, we found that, under Earth Reference System, when the ion thrusters' running time $t = 65452.66h$ (about 7.47 years), the total mechanical energy of the spacecraft is equal to E_0 (see below for its specific value). At this moment, close the ion thrusters, then the spacecraft will escape from Earth's gravity.

We plotted the trajectory of the spacecraft with sustained constant acceleration throughout the phase, as shown in the upper part of Figure 3. The portion of the trajectory of the spacecraft's numerous orbiting around the Earth is enlarged in the polar coordinate system as shown in the lower part of Figure 3.



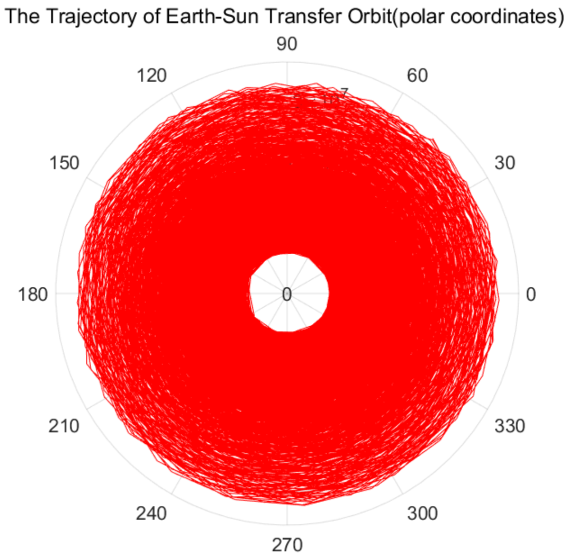


Fig3. The Trajectory of Earth-Sun Transfer Orbit

Figure 4 shows the relationship between the velocity of the spacecraft and time.

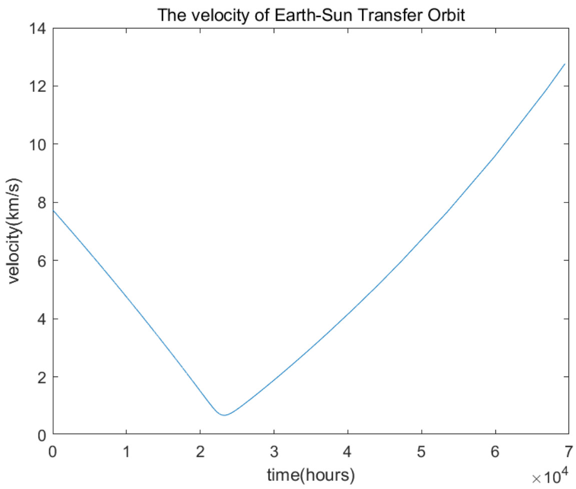


Fig4. The velocity of Earth-Sun Transfer Orbit

3.2 First Orbit Sun Phase

As the first basic assumption, we think the spacecraft is only exposed to Sun's gravity during the flight when it is free from Earth's gravity and has not been captured by Venus. Therefore it moves elliptically with Sun as its focal point.

At this moment, we establish a polar coordinate system with Sun as the polar point and the rays from Sun past the position of the spacecraft as the polar axis. Suppose all planets are in ideal position, thus the present velocity of the spacecraft can be decomposed into velocity along the axis pointing to the polar point v_1 and velocity perpendicular to the axis v_1', and the value of v_1' is exactly equal to the orbital velocity of Earth around Sun $v_E = 29.786km/s$.

When the spacecraft moves to the apex of the semi-long axis (the one close to the side of Sun), Venus moves to this position exactly. It then proceeds to the next model stage. So there is the following equation:

$$a_1 - c_1 = R_V \tag{2.1}$$

$$\mathrm{m} v_E R_E = m v_2 R_V \tag{2.2}$$

$$\frac{1}{2} m v_2^2 - \frac{G M_{SUN} m}{R_V} = \frac{1}{2} m (v_E^2 + v_1^2) - \frac{G M_{SUN} m}{R_E} \tag{2.3}$$

$$v_1 = 11.393km/s$$

$$v_2 = 41.179km/s$$

3.3 Venus Swing by Phase

We don't care about the process of the spacecraft's interaction with Venus, but only pay attention to its velocity when it happened to interact with Venus v_2 and its velocity when escapes Venus v_3 .

After this stage, the spacecraft goes directly to Saturn. To make sure that it can reach Saturn, we set:

$$a_2 - c_2 = R_V \tag{3.1}$$

$$a_2 + c_2 = R_S \tag{3.2}$$

From the equation and the properties of the conic curve itself, it can be solved that

$$v_3 = 47.759km/s$$

3.4 Second Orbit Sun Phase

From the above, it is known that the spacecraft flies from Venus along the conical curve described above to the vicinity of Saturn. Then it proceeds to the next stage.

$$\mathrm{m}v_3 R_V = \mathrm{m}v_4 R_S \tag{4.1}$$

$$v_4 = 3.6058km/s$$

3.5 Sun-Saturn Transfer Phase

Under Saturn Reference System, when the spacecraft orbits Saturn in a circle with an orbital period of 40 hours, $r_S = 2.7109 \times 10^5 km$, $v_5 = 11.829km/s$.

When the spacecraft is captured by Saturn, it firstly performs free motion under the gravity of Jupiter until the orbital radius is 300,000 km, then turns on the ion thrusters in the opposite direction of the angular velocity, slowing the spacecraft down until it reaches the prescribed orbit.

Use MATLAB to find the trajectory of the spacecraft's motion as shown in the figure 5.

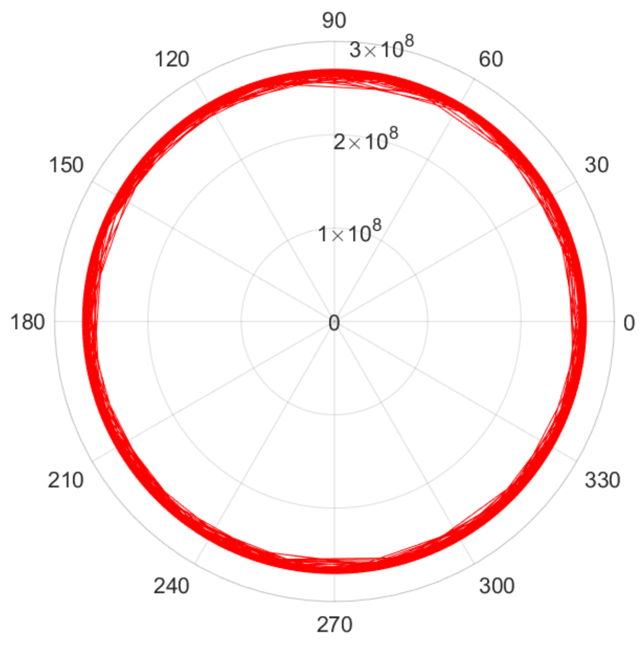


Fig5. The Trajectory of The Spacecraft's Motion

4 Results

During Earth-Sun Transfer Phase, according the above, the spacecraft takes 2402.75kg fuel and $t_1 = 65452.66\, h$.

During First Orbit Sun Phase, the spacecraft don't consume fuel. The time it takes:

$$z = \frac{R_E}{R_V} - 1$$

$$t_2 = \left[\cos^{-1} z - z\sqrt{1 - z^2}\right] \cdot \sqrt{\frac{\left(\frac{R_V}{1 - z}\right)^3}{GM_{SUN}}} = 1459.01 \text{ h}$$

During Venus Swing by Phase, the spacecraft don't consume fuel. Since its velocity in this process is very fast (>40km/s) and its trajectory is relatively short, we ignore the time it takes.

During Second Orbit Sun Phase, the spacecraft don't consume fuel. The time it takes:

$$t_4 = \pi \cdot \sqrt{\frac{(R_V + R_S)^3}{8GM_{SUN}}} = 51261.18\, h$$

During Sun-Saturn Transfer Phase, according the above, the spacecraft takes 0.02kg fuel and $t_5 = 0.43\, h$.

In summary, the minimum amount of fuel it takes is 2402.77kg. The duration of the trip is 4923.89 days (13.49 years). During Stage 1 ,turn on the ion thrusters in the same direction with the angular velocity for 65452.66 hours. While during Stage 5 , turn on the ion thrusters in the opposite direction of the angular velocity for 0.43 hours.

5 Strengths and Weaknesses

5.1 Strengths

1. With some basic assumptions and geometric relationships, we have greatly and reasonably simplified the construction of the model and the computation of the equations, and have made this voyage planning much more straightforward.

5.2 Weaknesses

1. Spacecraft is also affected by its satellites when orbiting the planet. Saturn, in particular, has a complex gravitational perturbation environment, making it difficult to

accurately predict the spacecraft's orbit. Orbital velocity calculated from the given orbital period may be subject to error.

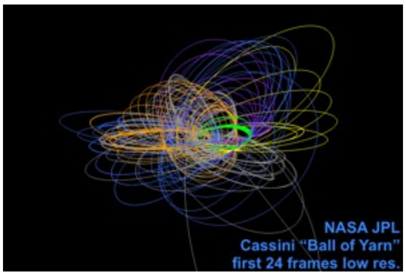


Fig6. Cassini “Ball of Yarn” first 24 frames low res.

2. This is only an ideal model. In the process of establishing our model, we think all the planets orbit the sun in circle and at a invariable velocity. In fact, there are many differences. We also ignore the effects of planets’ rotation and cosmic dust impeding the flight of spacecraft.

3. The orbital geometry relation we construct is only a rougher estimate. On the one hand, the dynamical feasibility is not considered deeply, on the other hand, the acceleration deflection angles during the interaction with planets are not included in the calculations.

MATLAB Code

```
function earth(a)
options = odeset('RelTol',1e-5,'Stats','on','OutputFcn',@odeplot)
  [t,y]=ode45(@odefun1,[0,5000000],[100000000;0;3000;-700],options)
  v=sqrt(y(:,3).^2.+y(:,4).^2);
  beta=atan(y(:,4)./y(:,3));
  figure(1);
  polarplot(y(:,2),y(:,1),'r')
  title('The Trajectory of Earth-Sun Transfer Orbit(polar coordinates) ')
  figure(2);
plot(t,v)
title('The velocity of Earth-Sun Transfer Orbit')
xlabel('time(hours)')
ylabel('velocity(km/s)')
figure(3);
plot(t,asin(a)+pi/2-beta)
title({'The Angle of Earth-Sun Transfer Orbit';'(Based on the vector of the vehicle.s path relative to the earth)'})
xlabel('time(hours)')
ylabel('angel(rad/s)')
    function dy=odefun1(t,y)
dy(1,1)=y(4);
dy(2,1)=(y(3)/y(1));
dy(3,1)=-0.400*double(a)/(2595.6-0.4/(9.8*4000)*t)-y(4)*y(3)/y(1);
```

```
dy(4,1)=0.400*sqrt(1-a^2)/(5000-0.4/(9.8*4000)*t)+y(3)^2/y(1)-(7736.8)^2*6648320
/y(1)^2;
     end
end
----------------------------------------------------------------------------------------------------
function saturn(r)
  format long

[t,y]=ode113(@odefun1,[0,30000000],[r;0;(3084.2^2+75862010230000000/r)^0.5;0])
  v=sqrt(y(:,3).^2.+y(:,4).^2)
  beta=atan(y(:,4)./y(:,3))
  figure(1);
  polarplot(y(:,2),y(:,1),'r')
writematrix([t,y(:,1),(y(:,3).^2.+y(:,4).^2).^0.5],'M.csv')
       function dy=odefun1(t,y)
dy(2,1)=(y(3)/y(1));
dy(3,1)=-0.4/(2598.25-0.4*t/(9.80665*4000))-y(4)*y(3)/y(1);
dy(4,1)=(y(3)^2/y(1)-75862010230000000/y(1)^2);
     end
end
```

Reference

[1]"Saturn Plunge Nears for Cassini Spacecraft". NASA - National Aeronautics and Space Administration. August 29, 2017. Retrieved August 30, 2017.

[2]Overbye, Dennis (September 8, 2017). "Cassini Flies Toward a Fiery Death on Saturn". The New York Times. Retrieved September 10, 2017.

[3] study on Trajectory Optimization Design Method for Multi-target DEEP Space Exploration. Li Jiu-tian.