Low-Thrust Orbit Transfer to Saturn with Spiral Raising

Problem A: Saturn Mission Analysis

Team 426

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Abstract

The use of ion thruster electric propulsion has seen considerable attention recently for its application in long-distance space travel. Electric propulsion achieves much higher fuel efficiency due to higher effective exhaust velocities and is therefore ideal for conserving payload space and minimizing fuel requirements. In this project, we analyze the parameters of the trajectory of an ion thruster-boosted spacecraft as it makes an orbit transfer from low Earth orbit to orbit around Saturn. We estimate the minimum fuel required to make a journey and the duration of the journey by dividing the trajectory into three regions, each of which are treated as a microthrust two-body problem: escape from Earth's orbit, orbit transfer from Earth to Saturn (where the Sun is treated as the central body), and capture into Saturn's orbit. In each region, spiral dynamics are explored and justifications are given for a fuel-minimizing trajectory. Time of flight, fuel required, and velocities are obtained for the boundaries at each region by numerically integrating their respective equations of motion. For the heliocentric region, in which the spacecraft traverses from Earth to Saturn, different thrusting strategies are investigated to determine a most fuel-efficient strategy. Finally, we estimate a fuel requirement of approximately 58.5% of the total mass and a total trajectory duration of 20.92 years.

Contents

Nomenclature

$\dot{m}$ Mass flow rate

μ Standard gravitational parameter of the Sun

ν_e Exhaust velocity

I_{sp} Specific impulse

m Mass

r Orbital radius

T Thrust

1 Introduction

The use of ion thruster electric propulsion systems for small probes and, in the future, manned spacecraft is of great interest. Long-duration ion microthrust engines have already seen application in missions like Deep Space and Dawn, which were able to achieve significant changes in velocity over long periods of time while using small amounts of fuel. Ion thrusters are able to achieve much greater specific impulses than traditional chemical rocket engines by expelling fuel at a much greater effective exhaust velocity [2].

In this project, we seek to analyze the trajectory of a small 5000-kg spacecraft using an ion thruster engine as it exits circular low-Earth parking orbit and enters a high circular orbit around Saturn by estimating the minimum fuel required and the duration of flight. A direct path is considered first, neglecting the gravitational field of bodies besides Earth, the Sun, and Saturn. Then, the effects of trajectory-optimizing techniques like thrust profiles are considered.

2 Background and Approach

We split the interplanetary trajectory of the spacecraft into three regions, each of which are analyzed separately: escape from low-Earth orbit, orbit raising within the Sun's orbit, and entry into Saturn's orbit. In each region, only gravitational interactions between the spacecraft and the central body are considered. This approach is commonly used to simplify optimization of interplanetary orbits and is similarly used here to simplify the analysis.

In all three regions, an analysis of spiral motion is required as a consequence of the very low acceleration. Orbit transfer can be simplified with the assumption of instantaneous impulses, resulting in optimization schemes such as the Hohmann transfer [4](Figure 1).

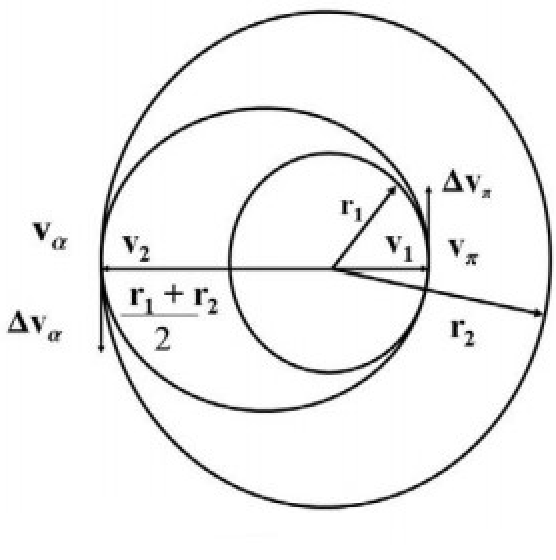

Figure 1: Hohmann transfer orbit with instantaneous impulses.

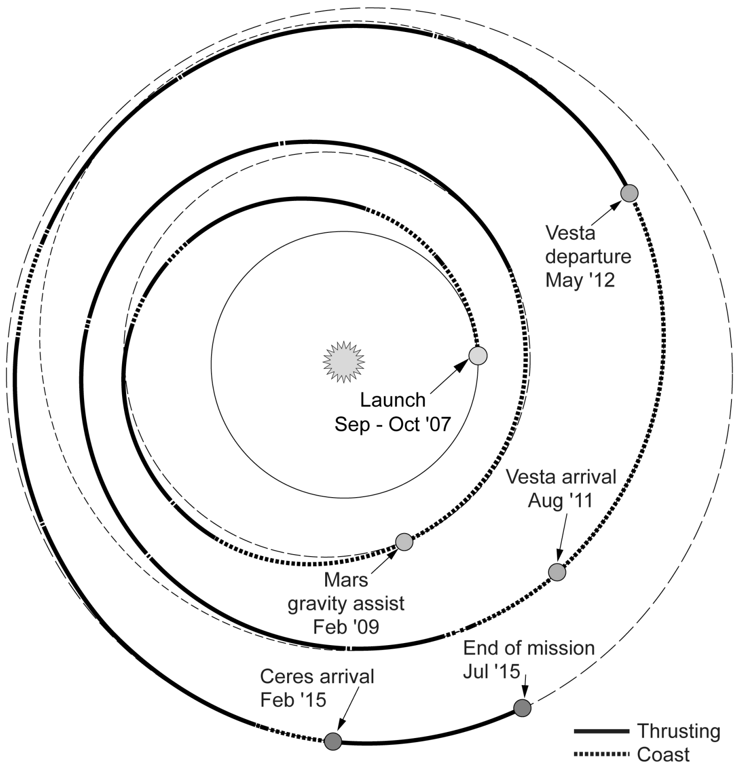

Figure 2: Dawn mission trajectory.

However, with low-thrust acceleration we can no longer make the assumption that impulses to the spacecraft are instantaneous. Rather, we consider the case of a small, continuous external force on the orbit. This results in a slowly increasing orbital radius as the thrust is applied to the spacecraft. For instance, the Dawn probe which traveled to the asteroid belt with microthrust ion propulsion engines traveled in an outward spiral trajectory in the heliocentric region (Figure 2).

For each region, an analysis of the spiral motion is done with the objective of finding the mass of fuel expelled and the time elapsed in that region. The weight of the fuel expelled per second can be computed from the specific impulse:

$$\text{weight/s} = \frac{\text{generated thrust}}{I_{sp}} \tag{1}$$

With $I_{sp} = 4000$ s, the change in mass follows immediately:

$$dm = -\frac{\text{generated thrust}}{g \times 4000 \text{ s}} dt \tag{2}$$

The negative sign arises because the mass is expelled from the spacecraft, so the change in mass is considered negative. As will be shown in the next section, the analyses of Regions 1 and 3 are done by numerically solving coupled ODEs in time, from which the duration of the trajectory follows immediately after specifying an end condition, such as a maximum radius or final energy level.

For regions 1 and 3, a numerical approach is taken to find the exit conditions by directly solving the differential equations of motion. A number of different approaches exist to describe the spiral trajectories of orbiting bodies under microthrusts. An analytic approximation can be made for small constant thrusts, which assumes the shape of the orbit lies very close to a circle. This approximation breaks down as the eccentricity of the orbit increases towards the exit [3] (Figure 3). Another technique which is used in low-thrust trajectory optimization approximates the continuous low thrust into a series of periodic instantaneous impulses [6]. This strategy would make numerical optimization of changing levels of thrust simpler, but is an unnecessary complication as we seek to estimate the fuel consumption and flight duration, not the exact optimal trajectory.

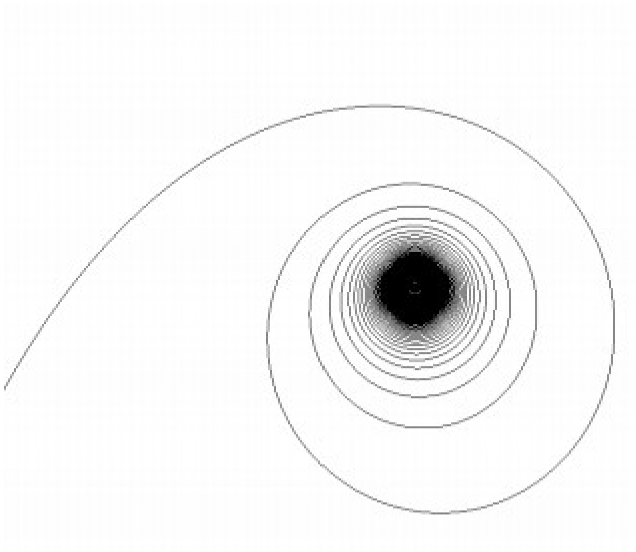

Figure 3: Example spiral orbit with increasing eccentricity.[5]

3 Analysis of Trajectory

In this section, the spacecraft trajectory is split into the three relevant regions, and separate analyses are done for each. For the trajectory in each region, we calculate the mass of fuel required, the time elapsed, and an ideal velocity as the spacecraft enters or exits the region. The initial conditions for Region 1 and exit conditions for Region 3 are taken from the orbit specifications in the problem statement, and the exit velocity for Region 1 and initial velocity for Region 3 are used as approximate boundary conditions for Region 2.

3.1 Region 1: Spiral Escape from Earth Orbit

First, the radius and velocity of the initial circular orbit are computed from the orbital period:

$$\frac{GM_E m}{r^2} = m\omega^2 r$$

$$r_0 = (\frac{GM_E}{\omega^2})^{\frac{1}{3}} = (\frac{GM_E T^2}{4\pi^2})^{\frac{1}{3}} = 6.65257 \times 10^6 \text{ m} \tag{3}$$

$$v_0 = \sqrt{\frac{GM_E}{r_0}} = 7740.6 \text{ m/s} \tag{4}$$

From this parking orbit, we assume the ion thrusters begin exerting a constant force of $F = 0.4$ N. We also assume the thrust is constrained to point in the direction of the velocity, i.e. $\mathbf{a} = \frac{F}{m(t)}\frac{\mathbf{v}}{v}$, where $m(t) = m_0 - \frac{F}{4000g}t$ in accordance with equation (2). We can then write the gravitational differential equation as

$$\frac{d^2\mathbf{r}}{dt^2} + GM_E\frac{\mathbf{r}}{r^3} = \frac{F}{m(t)}\frac{\mathbf{v}}{v} \tag{5}$$

Breaking down this equation into its radial and orbital components, we arrive at

$$\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2 + \frac{GM_E}{r^2} = \frac{F}{m(t)v}\frac{dr}{dt} \tag{6}$$

$$\frac{d^2\theta}{dt} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = \frac{F}{m(t)v}\frac{d\theta}{dt} \tag{7}$$

These coupled differential equations were solved with the numerical tool in Appendix A from the initial conditions set by the parking orbit. When the spacecraft exits Earth's sphere of influence, $r = 9.29 \times 10^8$ m, we calculate

$$v = 710.5 \text{ m/s}$$

$$m = 4169.9 \text{ kg}$$

$$\text{time elapsed} = 8.1402 \times 10^7 \text{ s} = 2.579 \text{ y}$$

Before moving on to an analysis of Region 2, it's worth inspecting how the assumptions we placed on the thrusts compare to the thrust of a fuel-minimizing orbit. First, we will justify the assumption of constantly maintaining the thrust at the engine capacity of 0.4 N. Recall that the mass difference Δm is proportional to the total impulse, which is proportional to the acceleration times the escape time. A careful analysis of the analytic approximation of the spiral escape yields [3]:

$$a_\theta t_{\text{escape}} \approx v_0(1 - (\frac{2a_\theta r_0}{v_0^2})^{\frac{1}{4}})$$

This expression rests on the assumption that the spiral orbits remain close to circular, so there is some uncertainty around the exact escape time t_{escape}. However, it does accurately describe the relationship where the product $a_\theta t_{\text{escape}}$ decreases for greater accelerations; thus the total impulse to reach escape conditions decreases with greater acceleration, and we are justified in assuming that the thrust stays at its greatest possible value.

Second, we inspect the assumption that the thrust is constrained to lie along the direction of velocity. Several analyses of optimized low-thrust spiral escape trajectories find that the fuel-minimizing trajectory has an applied thrust which lies not constantly parallel to velocity, but at an angle to the velocity vector which oscillates with time [5] [7]. However, optimized trajectories which do constrain thrust to the direction of velocity find an insignificant fuel penalty (0.13-0.39% increase in fuel consumption) compared to optimized, unconstrained trajectories [5].

3.2 Region 2: Orbit Raising to Saturn

Trajectories for the Earth-Saturn orbit transfer were calculated by numerically integrating the equations of motion for the spacecraft over time. Using a polar coordinate system centered on the Sun, the equations of motion in the spacecraft reference frame are [8]:

$$\dot{r} = v_r \tag{8}$$

$$\dot{\theta} = \frac{v_\theta}{r} \tag{9}$$

$$\dot{v}_r = \frac{v_\theta^2}{r} - \frac{1}{r^2} + \frac{cT\cos\alpha}{m} \tag{10}$$

$$\dot{v}_\theta = -\frac{v_r v_\theta}{r} + \frac{cT\sin\alpha}{m} \tag{11}$$

$$\dot{m} = -\frac{cT}{\nu_e} \tag{12}$$

Where v_θ and v_r are spacecraft's velocity in the tangential and radial directions respectively, m is the instantaneous spacecraft mass, and $alpha$ is the angle between the thrust vector and radial unit vector. For this analysis, $alpha$ was set to zero.

A constant $c = \frac{T_{max} r_0^2}{m_0 \mu}$ is introduced to normalize the initial conditions. Radii are normalized to 1 AU, mass is normalized to m_0, and velocities are normalized to Earth's orbital velocity.

This system of differential equations can be solved numerically to propagate the spacecraft's trajectory through time. The spacecraft's state at any point in time is represented by the state variable:

$$\boldsymbol{x} = [r, \theta, v_r, v_\theta, m] \tag{13}$$

To simplify the initial conditions, we assume that after exiting Earth orbit, the spacecraft is in a perfectly circular orbit around the Sun with Earth's orbital radius. In comparison to the vast distance between Earth and Saturn, the difference between the spacecraft's initial orbital radius and Earth's is small. This assumption allowed us to set $r_0 = r_{earth}$. $v_{r0} = 0$, and $v_{\theta 0} = v_{earth} + v_{escape}$.

The MATLAB program we wrote to solve the EOMs and generate plots of the spacecraft's trajectory is included in the appendix.

3.3 Finding Minimum-fuel Trajectories

Finding feasible, minimum-fuel trajectories is a complex optimization problem that is the subject of much research. To truly find the best trajectory, we would need to consider variable thrust profiles, thrust angles, launch dates, and gravity assist maneuvers. To constrain the problem and avoid the mathematical and computational complexity involved with such an approach, we instead aimed for a ballpark estimate of minimum fuel by considering classes of trajectories that can be obtained by only varying the thrust profile.

The first is the case of constant, continuous thrust. The second, continuous thrust that decreases with $1/r^2$. And finally, a case where the the engines are operated at maximum thrust for part of the journey, and then turned off. We found that the last approach consumed the least fuel and produced trajectories with reasonable transfer times.

For each case we have generated an example trajectory. These trajectories are not optimal solutions, and are included for illustrative purposes. The mass and time estimates are approximate. We noticed that varying the parameters of the trajectory slightly did not tend to have a large effect on fuel consumption, even if the trajectory had a very different shape. Furthermore, we did not impose boundary conditions on the trajectories. The goal was to obtain a trajectory that could be feasible for some arbitrary alignment of planets. In a later section, we will show that a launch window does indeed exist for the partial-coast trajectory.

3.4 Constant-Thrust Trajectory

The simplest case that we looked at was when we constantly applied the thrust. This lead to a logarithmic spiral as the spacecraft's loss of mass due to its fuel depletion lead to a constant force applied to the craft during the trip. This resulted in a sharp entry angle for the craft into Saturn's orbital path as shown in Figure 4. The fuel consumption was 50% of the initial spacecraft mass, while the duration of travel was about 8 years.

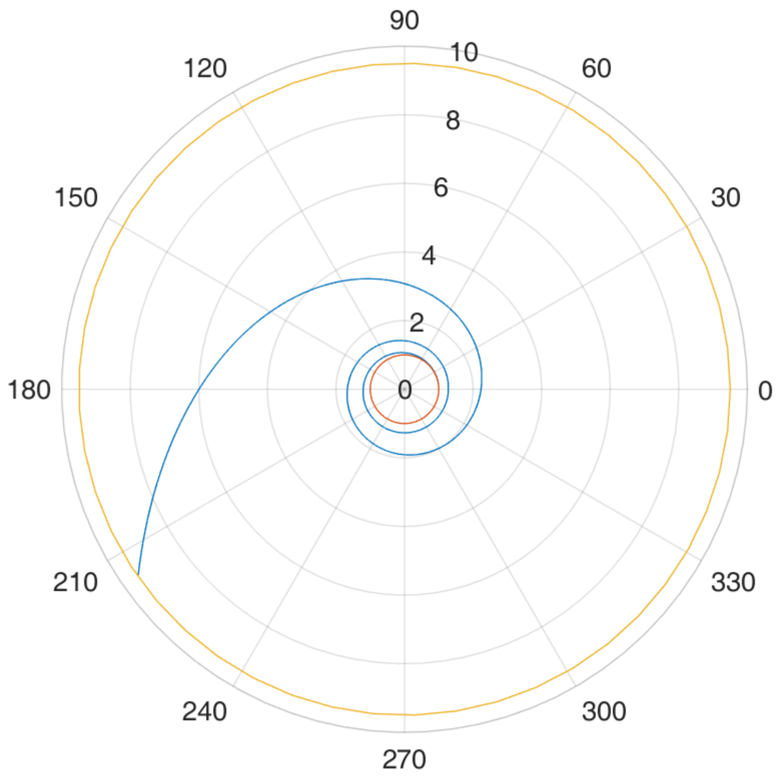

Figure 4: Constant-thrust spiral trajectory (distances in AU)

3.5 Decreasing-Thrust Trajectory

One strategy to minimize usage is to scale the intensity of thrust by $1/r^2$, where r is distance from the Sun. This also constrains the acceleration of the spacecraft as it loses mass, producing a more favorable Archimedian spiral which does not overshoot Saturn's orbit. Since the final velocity of the spacecraft will be in the direction of Saturn's orbit, less δV is needed for orbit capture.

Since the spacecraft engine only supplies a constant 400 mN of thrust, we would vary the thrust output by rapidly pulsing the engine. Varying the duty cycle would approximate an engine with variable thrust.

For the trajectory shown in 5, total mass usage was roughly 45% of the initial spacecraft mass, an improvement on the constant-thrust trajectory because the spacecraft spends less time operating at high thrust levels. However, the drawback of this approach is that it produces extremely an long and gradual ascent that takes decades to complete. As such, it is not the most practical approach for raising an orbit over such a large distance.

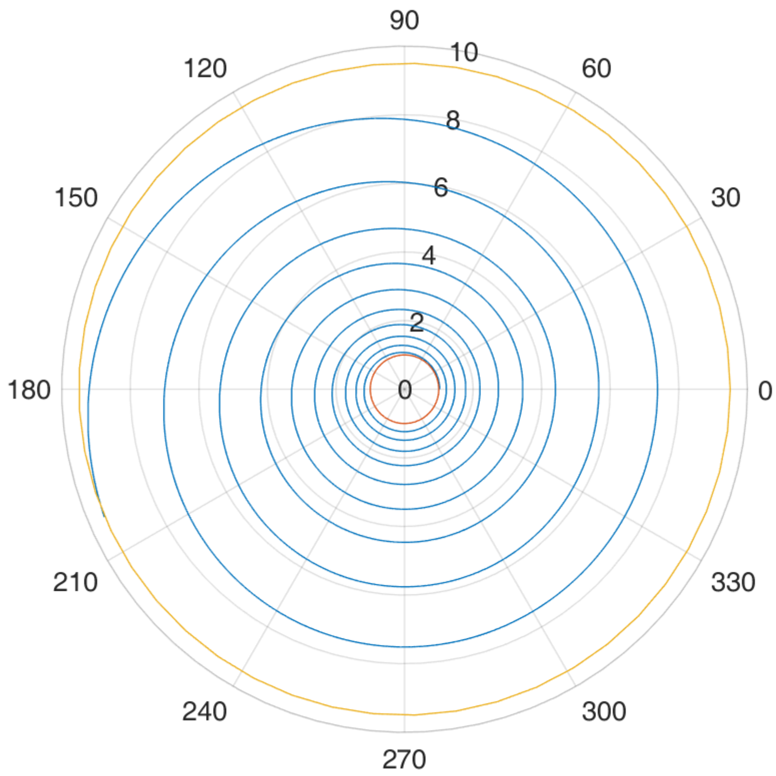

Figure 5: $1/r^2$ Thrust Trajectory

3.6 Partial-Coast Trajectory

The partial-coast strategy involves operating the spacecraft at maximum thrust for the first part of the journey, and then turning the engine off after a certain cuttoff distance r_{CT}. This approach takes advantage of the high engine thrust to quickly exit the inner solar system, but if tuned correctly, allows the spacecraft to be placed into an elliptical orbit once it has enough energy, which will intersect Saturn's orbit at it's apogee.

In the trajectory shown in Figure. 6, the engine was reduced to 50% output at 2.6 AU, and completely turned off at 2.66 AU, indicated by the cross on the diagram (note the the engine turns back on if the spacecraft orbit returns below 2.6 AU later on in the trajectory). This thrust regime produced the most favorable mass usage, only 32% of initial mass, and also a reasonable transfer time of 15.9 years. Another advantage is that the spacecraft reaches Saturn orbit with a much lower velocity than the other two methods, minimizing propellant usage in orbit capture. This orbit shape was used to generate an entry velocity for the spacecraft into Saturn orbit, treated in the next section.

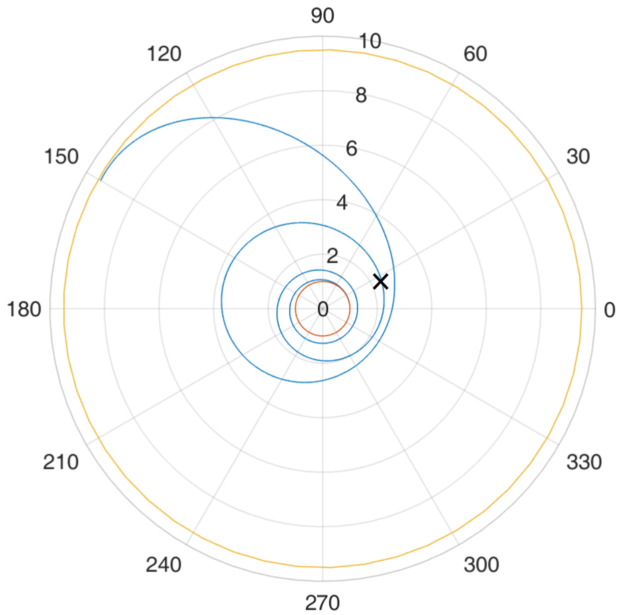

Figure 6: Trajectory where engines are turned off after 2.66 AU.

3.7 Region 3: Spiral Capture into Saturn Orbit

The radius and velocity of the final orbit around Saturn are computed with a similar calculation as in (3) and (4):

$$r_f = 2.71098 \text{ m}$$

$$v_f = 1182.9 \text{ m/s}$$

The analysis of the spiral capture trajectory is identical to that of the spiral escape, with the difference that the mass decreases as the spacecraft moves in toward the central body, rather than as the spacecraft exits the orbit. We treat the problem equivalently as that of a spacecraft exiting a circular orbit around Saturn with a thrust 0.4 N applied constantly in the direction of velocity as for Region 1, whose mass increases rather than decreases but at the same rate - effectively, we integrate the equations of motion backwards. The mass of the spacecraft in its final circular orbit is varied to match the spacecraft's entry mass to the final mass from Region 2. The entry point is taken to be where the total energy of the orbit obtained by integrating the equations of motion backwards, using the results of the partial-coast trajectory. With these constraints, we calculate

$$m_{\text{final}} = 2076 \text{ kg}$$

$$\text{time elapsed } = 7.649 \times 10^7 \text{ s} = 2.424 \text{ y}$$

3.8 Launch Window

Between late June 2038 and late December 2056 could be a potential window of time that the mission could take place. We know that the it will take roughly 2.58 years for the probe to escape Earth's orbit and another 15.92 years for it to intersect Saturn's orbital path. Setting the position of the Earth relative to the Sun (at launch) at zero degrees and taking into account Saturn's 29.457 Earth year orbital period, gives us that Saturn will move 62.8 percent along its orbit and 226.08 degrees since the start of our mission. Based on a rough estimate of our trajectory, the space craft will intersect Saturn's orbit around the 130 degree mark (still keeping the launch point as the zero degree reference). Using ephemerides taken for NASA's HORIZONS Web-Interface, late June 2038 was determined to be a possible mission start time, as 18.5 years from then Saturn will be close to the position of where the craft will enter its orbit [1].

4 Conclusions

The mass of the spacecraft when it enters the circular Saturn orbit is 2076 kg, so we estimate this trip would require a minimum of 2924 kg of fuel for the 5000 kg spacecraft. Summing up the duration of the trajectories in each of Regions 1, 2, and 3, we also estimate a total trip time of 20.92 years. These estimates are obtained by analyzing the trajectory taken by the spacecraft when it thrusts continuously in the direction of velocity to escape Earth's orbit, undertakes the partial coast trajectory as described in section 3.6, and then thrusts continuously in the direction opposite of velocity to slow down into a circular orbit around Saturn. This trajectory could be optimized by further optimizing the path taken in Region 2, as well as unconstraining the magnitude and direction of thrust in Regions 1 and 2. In addition, more complicated maneuvers such as gravity assists could further decrease the minimum required fuel and total trip time.

5 Appendix

5.1 Appendix A: C++ Numerical Tool to Calculate Earth Escape Orbit Parameters

```
include <iostream>
include <math.h>

static float F = 0.4;
static float GM = 3.98603e14;
static float g = 9.80665;
long double r = 6.652567399e6;
long double v = sqrt(GM/r);
long double m = 5000;


int main() {
long double dr = 0;
long double om = v/r;
long double d2r = F/m * dr/v + r*om*om - GM/(r*r);
long double dom = F/m * om/v - 2/r*dr*om;
double dt = 0.1;
double time = 0;
while (r < 0.929e9) {
r += dr*dt + 1/2*d2r*dt*dt;
om += dom*dt;
dr += d2r*dt;
v = sqrt(dr*dr + r*r*om*om);
m -= F/4000/g*dt;
d2r = F/m * dr/v + r*om*om - GM/(r*r);
dom = F/m * om/v - 2/r*dr*om;
}
}
```

5.2 Appendix B: Matlab Region 2 Trajectory Simulator

```
r0 = 1.495978770e11;
mu = 1.3271244e20;
v0 = sqrt(mu/r0);
tmax = 400e-3;
m0 = 4170;
c = tmax*r0/(m0*mu/r0);
ve = 4000*9.81;
alph = pi/2;

tspan = 0:0.001:200;
t,x
= ode45('propagate',tspan,x0);

polarplot(x(:,2),x(:,1))
hold on
earthorbit = linspace(0,2*pi,50);
polarplot(earthorbit, 1+zeros(size(earthorbit)))
saturnorbit = linspace(0,2*pi,50);
polarplot(saturnorbit, 9.5+zeros(size(saturnorbit)))
```

5.3 Appendix C: Matlab Region 2 Propagation Tool

```
function xdot = propagate(t, x)
r0 = 1.495978770e11;
mu = 1.3271244e20;
v0 = sqrt(mu/r0);
tmax = 400e-3;
m0 = 5000;
ve = 4000*9.81/v0;
alph = pi/2;
c = r0/(m0*mu/r0);

tmax = tmax/x(1)^2

xdot = [x(3) ;
x(4)/x(1) ;
x(4)^2/x(1) - 1/x(1)^2 + c*tmax*cos(alph)/x(5);
- x(3)*x(4)/x(1) + c*tmax*sin(alph)/x(5);
- c*tmax/ve];

end
```

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