Asteroid ocean impact and its relationship with the mass of the object.

Team 445

Problem B

Abstract

In this paper the relationship between an asteroid impact into water and mass of the asteroid was discussed. The main goal was to find the smallest mass of the asteroid whose impact could threaten remote coastal city. To achieve this goal the theoretic model was discussed. Because of the conclusion that the leading factor of causing damage to the city were tsunami waves, only their propagation was modelled and the model was also discussed. The smallest possible asteroid mass for generating tsunami waves about 10 metres was estimated to be about 10^{10} kg. Such mass corresponds to an asteroid with radius of about 97 metres. To show the scale of damage caused also range of the tsunami was calculated. This value was estimated at about 715 metres. These results have been considered to be possible. Possible ways to increase accuracy of the model were also discussed.

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1 Introduction

Asteroid impacts are one of the most hazardous astronomical phenomena that can nowadays happen on Earth. We can divide them into two categories: an impact into land and an impact into ocean. Each of them requires different theoretical approach and needs different theoretical model to predict what happens during and after the impact. In this paper we will only discuss marine impacts and their effects, such as generation of air blast waves, tsunami waves and seismic waves. We will also discuss how these impacts can threaten coastal cities.

2 Problem analysis

The main task is to estimate the smallest mass that can cause significant damage to a coastal city. For this problem the city is about 1000 km from the impact site. Firstly, a good idea is to simulate, how much kinetic energy would the asteroid transfer to environment upon the impact. Therefore we need to calculate kinetic energy loss during traversing through the atmosphere and from that, the velocity of the asteroid upon impact.

Knowing approximately how much energy was transferred to air, water and the ocean bed, we can simulate propagation of waves generated due to the impact. We will base it on existing models of wave propagation in each medium. Having working simulations, it is possible to obtain physical parameters of these waves.

Then, physical parameters of these waves will be used to estimate factors, such as height of the tsunami wave, energy of the air blast wave and amplitude of seismic waves, which are direct cause of destruction of the city. Given that, we can define what is a substantial damage in that case. By substantial damage we can assume damage made by tsunami wave about 10 metres high, blast wave that destroys smaller building or seismic wave with magnitude of about 7 Richter. Each of these factors has been defined based on destruction caused by each of these waves in the past. For the seismic and the blast wave we estimated value based on appropriate tables, for the tsunami wave, based on historical data, we estimated that wave 5 metres high is high enough to cause substantial damage to building and harm people inside. For the tsunami wave we can also estimate how deep inland it can go.

Given the conditions and physical parameters of waves dependent on mass of the asteroid we can find smallest mass for which one of these conditions will be met.

3 Made assumptions

To compute a simulation in a really short time, a lot of assumptions is needed to be made. Most of them has its own explanation in the Theoretical model section. All of them are listed below:

- the asteroid is a spherical body up to impact

- the asteroid motion is perpendicular to Earth
- changes of gravitational field are negligible
- in the best case scenario fragmentation doesn't occur
- Earth's curvature is neglected
- the deep ocean bed is flat and has constant depth
- density of water is uniform
- density of air in Earth's atmosphere changes exponentially
- impact cavity is radially symmetrical and has parabolic cross section
- impact cavity diameter to depth ratio is fixed
- tsunami wave propagation is cylindrically symmetrical
- blast wave propagation is spherically symmetrical
- heat radiation impact on the city is negligible
- ocean water is an incompressible medium

4 Theoretical model

4.1 Atmospheric entry

In this section we provide theoretical model for asteroid's motion from the moment of entering atmosphere up to the impact with the ocean surface. In particular we derive how mass, size and velocity of the projectile change during that process.

We assume that the asteroid is a spherical body with initial mass m_0, constant density ρ_{ast} and its motion in Earth's atmosphere is perpendicular to Earth's surface.

4.1.1 Forces acting on the asteroid

During the atmospherical entry there are two forces acting on the body: drag force F_d and gravitational force F_g, which are described by equations

$$F_d = \frac{1}{2} C_d S \rho_{atm} \dot{x}^2, \tag{1}$$

$$F_g = -mg \tag{2}$$

where m, S and x are mass, cross section area and height above the ocean of the projectile, g is gravitational acceleration, ρ_{atm} is density of the atmosphere and C_d is drag coefficient of the asteroid. Changes in g are negligible in our region of interest (Earth's atmosphere), so we will assume that g is constant.

From Newton's second law we derive the equation of motion of the projectile

$$\frac{d}{dt}(m\dot{x}) = \dot{m}\dot{x} + m\ddot{x} = -mg + \frac{1}{2}C_d S \rho_{atm} \dot{x}^2, \tag{3}$$

where after substituting $\eta = \frac{1}{2}C_d S \rho_{atm}$ for simplicity we get

$$\frac{\dot{m}}{m}\dot{x} + \ddot{x} = -g + \frac{\eta}{m}\dot{x}^2. \tag{4}$$

In order to solve this differential equation we need to find relation between $\dot{m}$ and x, which is obtain in the next paragraph.

4.1.2 Ablation

In order to account for mass loss due to ablation, let's consider mechanical energy Q of the asteroid in a given moment

$$Q = \frac{1}{2}m\dot{x}^2 + mgx. \tag{5}$$

This energy is converted into heat with rate

$$\dot{Q} = \dot{m}\left(\frac{\dot{x}^2}{2} + gx\right) + m\dot{x}(\ddot{x} + g). \tag{6}$$

Now, notice that during ablation some vaporised mass $\dot{m}$ detaches from the asteroid keeping its mechanical energy $\dot{m}\left(\frac{\dot{x}^2}{2} + gx\right)$. Remaining energy is the energy needed to vaporise this mass, which is described by the second part $m\dot{x}(\ddot{x} + g)$, so we get

$$C_p\dot{m} = m\dot{x}(\ddot{x} + g) \tag{7}$$

where C_p is heat of vaporisation of the asteroid material.

4.1.3 Solution

Substituting $\dot{m}$ from equation (7) into equation (4) and solving for $\ddot{x}$ gives us following formula

$$\ddot{x} = -g + \frac{\eta\dot{x}^2}{m(1 + \frac{\dot{x}^2}{C_p})} \tag{8}$$

This allows us to rewrite equation (7) for mass loss due to vaporisation in explicit form

$$C_p\dot{m} = m\dot{x}\frac{\eta\dot{x}^2}{m(1 + \frac{\dot{x}^2}{C_p})} = -\frac{\eta\dot{x}^3}{1 + \frac{\dot{x}^2}{C_p}}. \tag{9}$$

Obtained formula agrees with results from [1] and generalises that result in case, when the mass lost during the entry is not negligible compared to the initial mass of the asteroid. We will use the formula (8) to calculate the final velocity of the impactor.

4.1.4 Fragmentation

During traversing through the atmosphere due to differential atmospheric pressure, fragmentation can occur. Atmospheric pressure on the leading face is much greater than on the tailing face when the pressure on the leading face exceeds characteristic strength of the material. In this process the asteroid is fragmented into smaller objects and because of that ablation rate and deceleration increases and it is less likely for the asteroid to reach the surface and energy dissipates only in air. Fragmentation doesn't occur for objects with diameter much greater than 1 km. In our case we assume best case scenario – fragmentation doesn't occur and the asteroid reaches the surface [1] which allows the biggest amount of energy to dissipate in water, air and the ocean bed.

4.2 Impact

In this section we provide theoretical model of the impact of the asteroid with the ocean surface. The model is based on cited papers [2] and [3]. We also won't discuss heat radiation emitted during impact, because it dissipates really quickly into water and if it should affect the city in any way, first it would have to vaporise all of the ocean on its way.

4.2.1 Energy transfer

The impact velocity of the asteroid is for most cases big enough for the ocean surface to behave like a solid and thus for asteroids impact velocities moment of the impact into water looks in a big approximation like the impact into land. Due to the impact air blast wave is generated because of rapid increase of air temperature and pressure. Upon the impact an ocean cavity forms instantaneously, which collapses moments later and generates tsunami waves. Then the asteroid traverses through water and is slowed down by the water drag force. The asteroid losses its kinetic energy which is transferred to water and amplifies generated tsunami waves. Upon hitting the ocean bed the asteroid generates seismic waves and, if it has enough velocity, even a crater [3].

4.2.2 Tsunami wave

As was previously stated, upon hitting the ocean surface the asteroid creates impact cavity. As proposed by Ward [2] the cavity for the moderate size asteroids has a radially symmetric, parabolic shape of depth D_C and inner and outer radii R_C and R_D. After Ward [2] we will assume that $R_D = \sqrt{2}R_C$, because in that way we minimise energy loss due to splashed water not contributing to tsunami waves generation. In this case the formula for D_C is

$$D_C = \sqrt{\frac{2\epsilon\rho_{ast}v_I^2R^3}{\rho_w g R_C^2}} \tag{10}$$

where ϵ is a fraction of the kinetic energy of the asteroid which contributes to generating the tsunami, v_I is the impact velocity of the asteroid, R is its radius and ρ_w is the density of water. According to Ward [2], ϵ holds rather constant and its value is $\epsilon \approx 0,15$. That value was calculated based on data from tests conducted with explosives which simulated the asteroid impact. Following the data from experiments shown by Ward [2], we can make simplification of fixing the diameter to depth ratio. Equation (10) then reduces to

$$D_C = \frac{d_C}{3} = \left(\frac{8\epsilon\rho_{ast}v_I^2}{9\rho_w g}\right)^{\frac{1}{4}} R^{\frac{3}{4}} \tag{11}$$

where d_C is the diameter of the cavity. It is also a good idea to consider what would happen if the depth of the cavity was greater than the depth of the ocean bed. For a deep ocean that would happen only for really massive asteroids. In this scenario the depth of the real cavity equals depth of the ocean bed and the crater in the ocean bed is made, as suggested in [3].

4.2.3 Air blast wave

Air blast wave is generated at the moment when the asteroid hits the ocean surface. About 15% to 20% of kinetic energy of the asteroid contributes to air blast wave generation [5]. First, we want to only estimate wave intensity near the city. We do this to determine which wave would cause substantial damage to city for the smallest possible mass. For air blast wave parameter that we want to estimate is its intensity. For this simple estimation we assume that almost none energy is lost during propagation, propagation is spherically symmetrical and waves reflected from the ocean surface don't affect the rest. We can assume that because we want only estimation which doesn't have to be as accurate as possible. Because of propagation being a 3 dimensional problem the air blast wave intensity changes like $\frac{1}{r^2}$. According to Ward [2] we know that the amplitude of the tsunami wave, which is its parameter defining destructive capabilities, changes like $\frac{1}{\sqrt{r}}$. Because energy which contributes to generation of both waves is similar, we can conclude that for greater distances, like 1000 km, tsunami waves will have bigger impact on the city than air blast waves and thus we don't need to model their propagation accurately.

4.2.4 Seismic wave

Seismic waves are generated when the asteroid hits the ocean bed. Because of most energy being transferred to water and air earlier, only a small fraction of impact kinetic energy contributes to seismic waves generating. According to calculations less than 1% of kinetic energy is transformed into seismic waves energy [5]. Considering that we can deduce that seismic waves will play minor role in our simulation, because their energy is much smaller than that of the other waves.

4.3 Propagation of tsunami waves

Without loss of generality we can assume that propagation of tsunami waves is cylindrically symmetrical and we can reduce the wave propagation to a 2 dimensional problem. In general we need to solve system of nonlinear differential equations when we encounter this type of problem, but instead of that we will try to simplify problem using linearised set of equations. In our solution we will try to find a solution for set of basic initial functions and then compose our general solution as linear combination of basic solutions. Following Ward in his lecture [4] where he derives formula for ocean surface elevation from wave equation, elevation as a function of distance, depth and time is given

$$u(x,0,t) = Re\left(\int_{-\infty}^{\infty} dk \frac{u_0(k)}{2\pi} \exp(i(kx - \omega(k)t))\right) \tag{12}$$

where $u_0(x)$ are elevation initial conditions, k is a wave number, $\omega = 2\pi f$ and f is a wave frequency. In Ward's derivation of formula (12) one crucial assumption has been made. We assume that ocean bed has uniform depth, which allows us to neglect small changes in depth that could cause disturbances in wave propagation which could highly complicate simulation, but wouldn't affect our estimations in any significant way.

Ward [2] also derives from equation (12) the formula for maximum height (amplitude) for a given distance r from the impact site and depth h. It is given by

$$u(r,h)_{max} = D_C \left(\frac{1}{1+\frac{r}{R_C}}\right)^{\left(0{,}5+(\gamma_1-0{,}5)\exp(\gamma_2 \frac{R_C}{h})\right)} \tag{13}$$

where γ_1 and γ_2 are constants calculated by fitting the function to the experimental data. According to Ward [2] their values are $\gamma_1 = 1{,}075$ and $\gamma_2 = -0{,}035$.

Additionally, we can assume that ocean water is incompressible, which effectively says, that we neglect all the overtones, as described by Ward [4]. That leads to dispersion equation for gravitational waves in deep water

$$\omega^2 = gk \tanh(kH) \tag{14}$$

which can be solved numerically both for $\omega(k)$ and $k(\omega)$ and used in equation (12).

As the wave propagation formula (12) assume constant ocean depth, it doesn't hold for the continental shelf, where the depth changes rapidly. Hence, we have to use another model for that part of motion.

While research was conducted, another tsunami wave propagation model has been found. It is called shallow water wave model and is based on similar assumptions as Ward's model [6]. For a significant distance from the impact site, where wave propagation stabilises, tsunami wave propagation behaves really similar to the model used in this paper. However it doesn't account for effect of the impact on initial tsunami wave amplitude and velocity. Because of that we decided no to use that model in our estimations.

4.4 Run-up of tsunami wave

When the tsunami wave approaches the continental shelf, Ward's wave propagation formula (12) starts to overestimate wave's height. That part of movement of the tsunami wave is crucial to modelling how hazardous it is as in this part the process called run-up occurs. The run-up is a rapid increase of wave's amplitude at expense of wave's velocity. Because water wave's energy depends on its amplitude and ocean bed depth, given the approximation that the wave's energy is conserved, which holds true in most cases, with decrease of depth, wave's amplitude raises [2]. On the other hand wave's kinetic energy depends on velocity, so if mechanical energy is constant and amplitude raises, the wave has to slow down. Ward [2] approaches the problem in a similar manner and using Green's theorem he estimates tsunami height as

$$H = A^{\frac{4}{5}}(r)h^{\frac{1}{5}}(r) \tag{15}$$

where we take amplitude A and depth h at any near shore distance where $A < h$. Moreover we can estimate how deep inland such a wave can go. Following Ward's estimation [7], assuming that the land is flat, we are given formula for the inland distance

$$X(R, H) \approx 10\sqrt{gH}(2R)^{\frac{3}{8}} \tag{16}$$

5 Simulation

5.1 Atmospheric entry

Equations (8) and (9) form a system of differential equations. With given initial values m_0, $v_0 = \dot{x}(0)$ and $x_0 = 100$km (which is commonly agreed height of Earth's atmosphere) we evolve values of m, x, $\dot{x}$ using Runge-Kutta method implemented and calculated in Julia programming language.

To account for aerodynamic drag a simple model of isothermic atmosphere is used (ref), in which density of air changes exponentially with height

$$\rho_{atm}(x) = \rho_{sea} \exp\left(-\frac{x}{x_K}\right) \tag{17}$$

where $\rho_{atm} = 1.3\frac{\text{kg}}{\text{m}^3}$ is atmosphere density at sea level and $x_K = 10$km is an empirical constant.

We use heat of vaporisation expected for iron and stony asteroids $C_p = 8 \cdot 10^6 \frac{\text{J}}{\text{kg}}$ [1] and drag coefficient typical for a rough sphere of $C_x = 0.5$.

5.2 Deep water propagation of tsunami waves

In order to solve our problem we will use sampling to represent our initial situation as set of initial solutions. Simulation of evolution of $u(x, 0, t)$ from equation (12) is possible by using Fast Fourier Transform to find coefficients $u_0(k)$. Then, each coefficient evolves independently following the equation

$$u_0(x, t, k) = u_0(k)e^{i(kx-\omega(k)t)} \tag{18}$$

where k and ω are related by equation (14) and after time t_i we combine the functions $u_0(x, t_i, k)$ using Inverse Fast Fourier Transform to get results. In reality we approximate u_0 by a discrete function $A[x]$ and use discrete forward and inverse transforms. This allows to get credible results in reasonable time.

Notice that solution is assumed to be periodical, so in order to get proper result for our problem we must solve the equation when all interesting phenomena happen near small x compared to the domain. We can use our simulation to learn more about propagation of the waves in the ocean. In all of our simulations we used model with ocean depth $H = 4$ km. As we can see on the plots typical tsunami wave is composed from various waves with different pulsations. Because our dispersion relation is not linear, waves evolve with different velocities what we can easily see on Figure 2. We can also confirm, that maximal amplitude changes with distance from impact place r like $\frac{1}{\sqrt{r}}$, what was postulated previously.

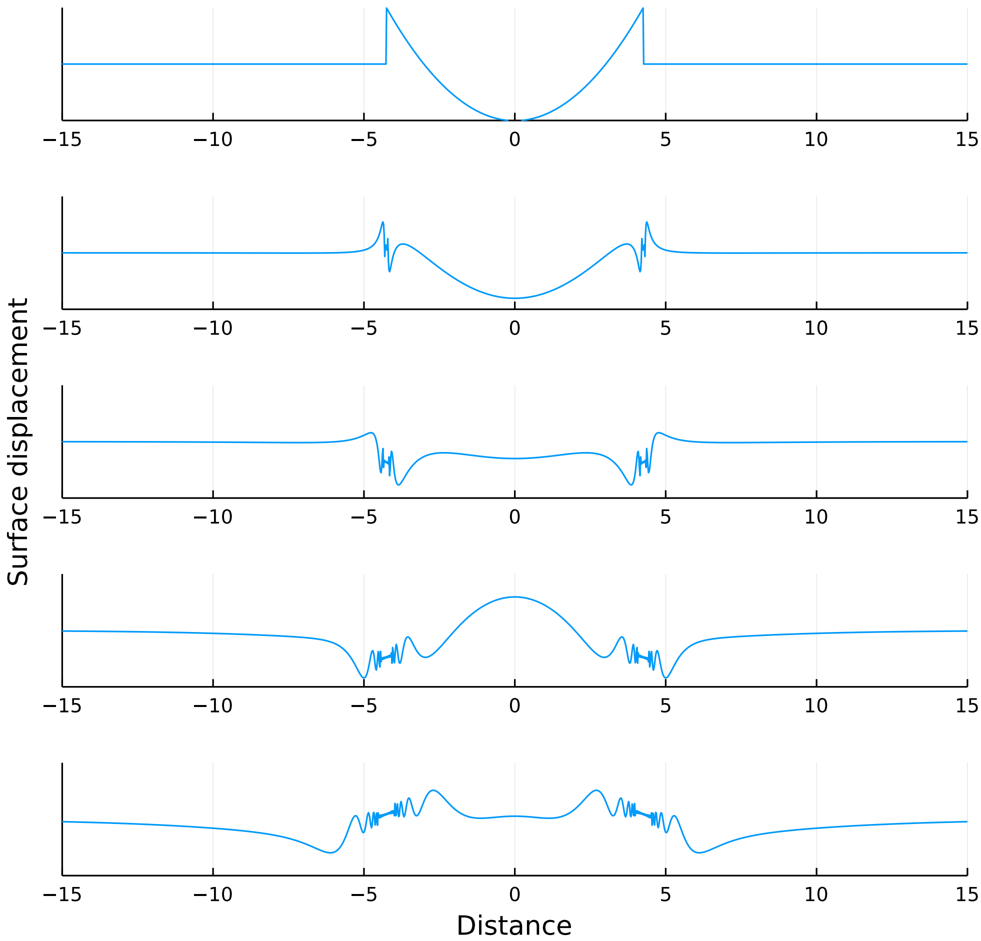

Figure 1: Propagation of tsunami wave after moments after impact of asteroid with mass $m = 10^{10}$kg. Timestep between subsequent plots is 10 s. For simulation we used 10^5 points equally spaced over $(-1000, 1000)$ range.

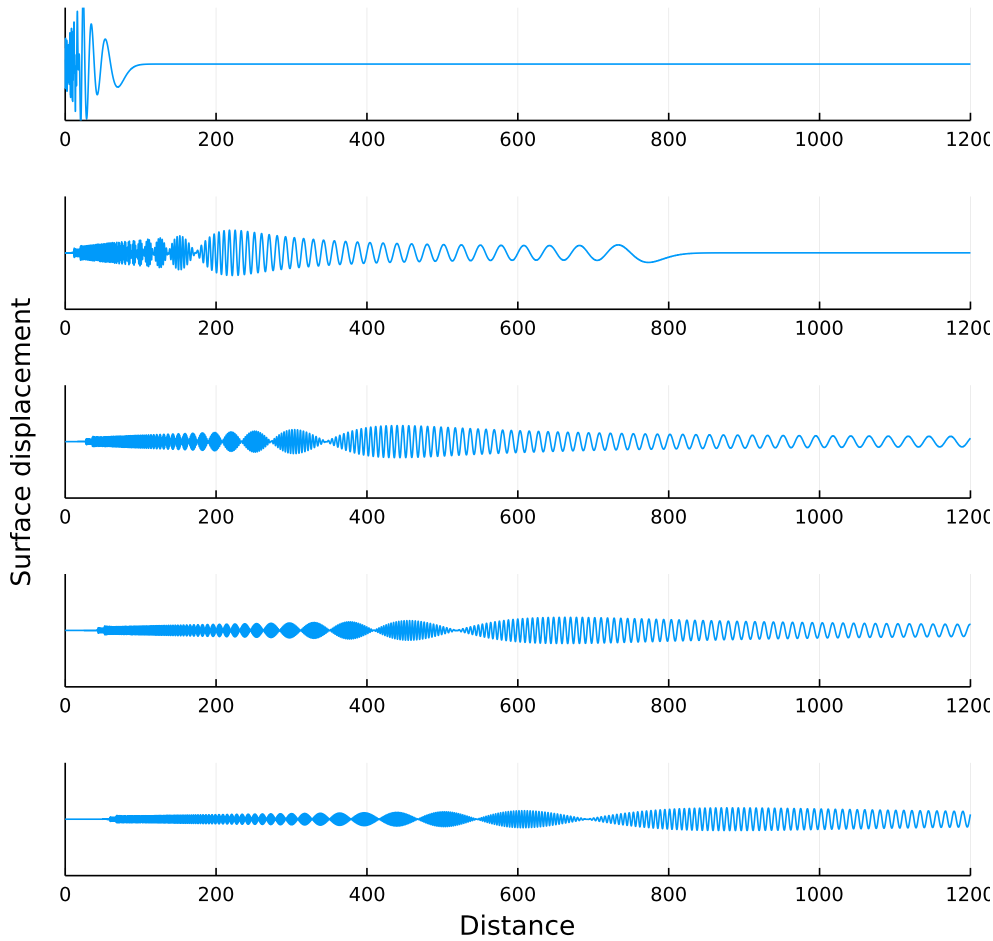

Figure 2: Propagation of tsunami wave after impact of asteroid with mass $m = 10^{10}$kg. Timestep is 4000 s. Uses $5 \cdot 10^5$ points equally over $(-5000, 5000)$ range.

5.3 Run-up of tsunami wave

Earlier we assumed uniform ocean depth far from the coast. Around 5 kilometres from the coast there is rapid depth decrease as we reach the continental shelf. The slope of the continental shelf is about $0,1°$ and in the deepest point it is about 500 metres deep [8]. At this point the amplitude of the tsunami wave starts increasing following (15). To calculate run-up wave height we can choose any point on the continental shelf, so we will choose the point at the begging of the shelf with $h = 500\,\text{m}$. Knowing wave height we will calculate how far inland the wave will go and we will find the mass that generates wave height of about 10 metres.

6 Results discussion

First thing to notice is that traversing through the atmosphere, as we expected for high velocity objects, does not really affect object's mass or its speed. With initial velocity of about 22 $\frac{\text{km}}{\text{s}}$ (which is the value for most of the asteroids in the Solar System that could hit the Earth), its impact velocity was calculated to be about 20 $\frac{\text{km}}{\text{s}}$. Example plot of mass, height, and velocity during entry can be seen in the Figure 3. As we can observe, for such high velocity object the change in velocity and mass are negligible.

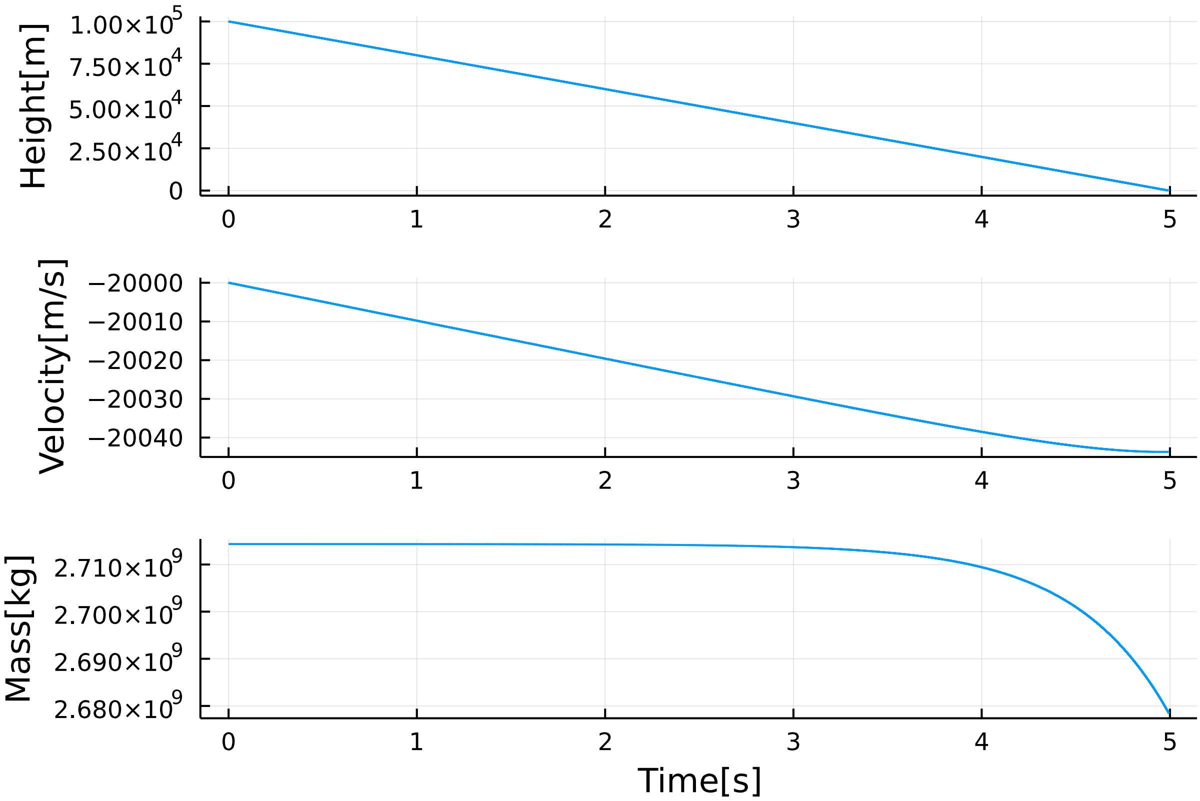

Figure 3: Examples of plots of velocity, height and asteroid's mass as a functions of time during its travel through the atmosphere.

The maximum wave height calculated from the equation (13) as a function of mass at fixed distance of 995 km and depth of 5 km can be seen in the figure 4. As expected the function has the asymptote. Knowing the function of the amplitude we calculate the tsunami height as a function of mass. The function is shown in the figure 5. As we can see the wave height of about 10 metres is produced when the asteroid has mass of about 10^{10} kg. Such mass corresponds to the asteroid's radius $R \approx 97\,\text{m}$. We also calculate how far inland our tsunami wave will go. For tsunami height of about 10 m and the asteroid's radius of about 97 m, the distance travelled by the tsunami is $X \approx 715\,\text{m}$. Tsunami waves of about 10 metres can destroy smaller building, kill people who didn't find shelter and displace cars and other objects of similar weight. So we can finally conclude that the smallest asteroid mass for which ocean impact will cause substantial damage is about 10^{10} kg.

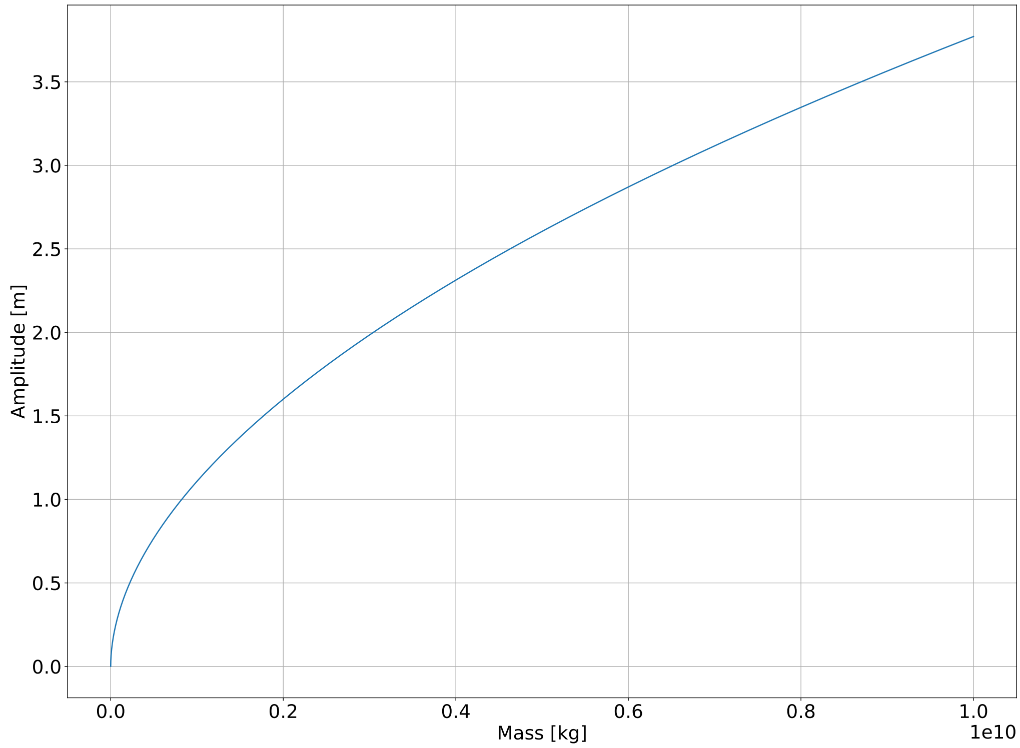

Figure 4: Amplitude of the tsunami wave at fixed distance of 995 km and depth of 5 km as a function of mass.

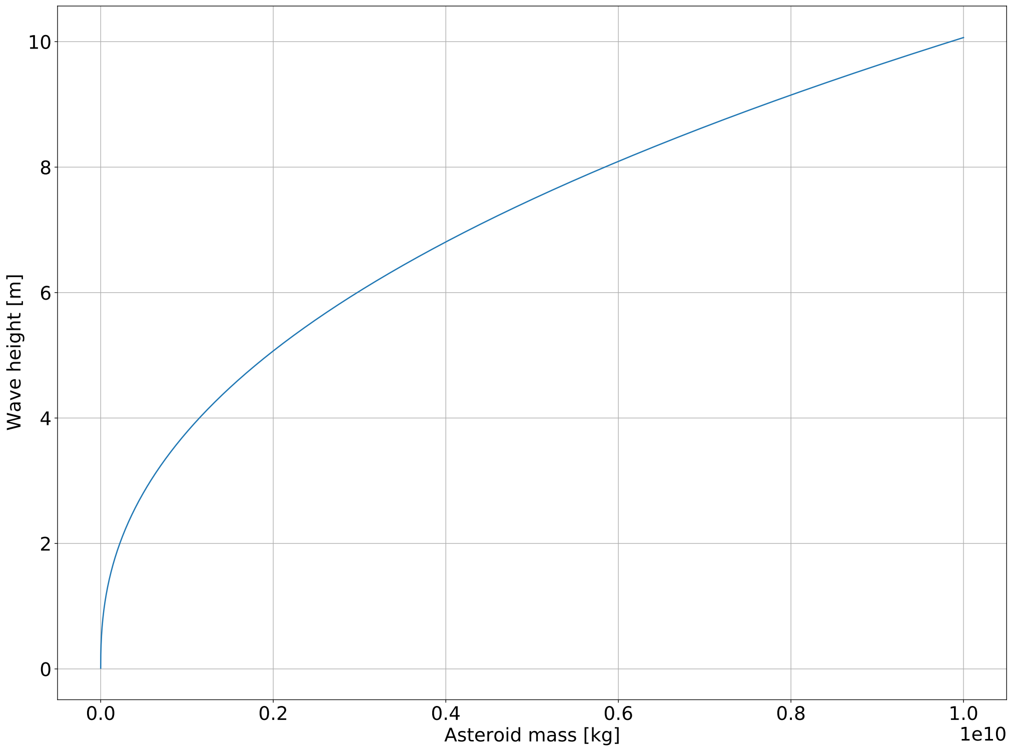

Figure 5: Tsunami wave height near the shore as a function of mass.

7 Summary

While building our theoretical model, we made a lot of assumptions that simplified calculations and rendered the problem possible to simulate in a rather short time, but on the other hand because of these assumptions and simplifications our answer is quite inaccurate. To increase accuracy we should either create better numerical model of wave propagation or use simulations called hydrocodes. Unfortunately neither of them could be computed during competition so we opted for inaccurate and quantitative solution which has given us only an estimation of the exact value.

There is still a lot of work to be done to find the mass accurately. Moreover during research we found other models of water wave propagation, asteroid impact into the ocean and energy transfer during the impact. A good idea would be to check these different models and compare them with the one used in this paper.

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