Deflecting an asteroid

Problem A

Team 425

Abstract

In our work we considered an asteroid with a diameter of 100 m on a trajectory to directly impact the Earth at a speed of 25 km/s. With proper assumptions we managed to parameterize with one parameter – impact angle orbits that such asteroids can have. We restricted ourselves to asteroids that are bounded to Solar System. Further we considered the disturbance of asteroid's trajectory after adding certain momenta to it (collision with spacecraft), we found the optimal way of momenta transfer to be: to hit asteroid, parallel to its own velocity, in the asteroid's perihelion. We found also that disturbance in asteroid's period is greater than disturbance of its orbit's axes, and we focused on the first value neglecting the later. We estimated value of momenta that we can transfer to asteroid with our spacecraft. Having that we calculated the disturbance of the period for 20 different asteroid's orbits (characterized by impact angle). Further we obtained condition on minimal time difference, that has to result from spacecraft's collision, to prevent Earth-asteroid collision. That together with values of orbit's periods allowed as to get final results. We obtained that we have to hit asteroid between 0.1 to 0.7 year before its possible collision with Earth.

1 Introduction

Let us consider space objects posing a possible threat to earth (i.e. having trajectories which intersect the Earth's orbit) we can divide them into two categories. One category will contain asteroids bound to our solar system, while the second one - comets with hyperbolic trajectory (or effectively hyperbolic, as seen in the proximity of the Sun). According to data gathered by NASA Center for Near Earth Object Studies, a vast majority (>99.57 %) of objects considered as the threat, are asteroids with closed orbits around sun. This can be easily understood in the face of the fact that the hyperbolic object would have the chance to hit the earth only once during its flight through the Solar System, while the asteroids closer to the Sun can cross Earth's orbit every orbital period. impactFamous recent event of a Chelyabinsk meteor entering the Earth's atmosphere on 15 February 2013 and causing injuries to several people and vast property damage was related exactly to this kind of asteroid. We have decided to exclude the objects with hyperbolic trajectories from our model, and focus on objects bound to Sun. class

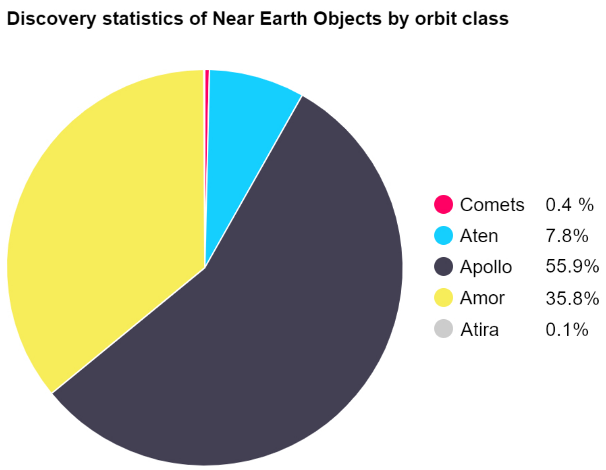

Figure 1: Discovery statistics of Near Earth Objects according to NASA CNEOS. Most comets and all four asteroid classes correspond to closed orbits around Sun

2 Numerical simulations

The main tool for dealing with the problem is using numerical simulations. In order to solve the problem we use Verlet integration [5] method. This approach gives better results than the easiest Euler's method, because the algorithm is time-reversible, just like the physics laws.
The algorithm was implemented in programming language *Python*. During simulation we do small time-steps dt. At each step we calculate new gravitational force acting on the object and its new position. The value of dt depended on the strength of the force. In the close-to-Earth locations the value was $dt \approx 10$s, but in the larger distances it became $dt \approx 200$s.
In order to calculate the change of period of asteroid orbit caused by collision with spacecraft we numerically calculated the new semi-major axis through numerically looking for the closest and most far positions from the Sun.

3 Restatement of the problem and a model used for obtaining results

3.1 Asteroid's trajectory

We consider a class of objects defined as "an asteroids on a trajectory to directly impact the Earth at a speed of 25 km/s". To see what orbits correspond to the class of such objects we reversed the problem and analyzed a trajectory of an object starting from Earth's surface with velocity (in the Earth's frame of reference) $v_{col} = 25\frac{\text{km}}{\text{s}}$ at the angle ϕ, in the radial direction. In this way we parameterized by an angle parameter ϕ all possible orbits belonging to the class. Here we make two reasonable assumptions. One is that asteroid's orbit lies in the same plane as Earth's (we use only an angle ϕ) which is the case. The other one is that Earth's orbit is ideally circular - we do not consider different points (with respect to Sun) of collision but assume that there is a symmetry. The assumption is quite reasonable, as Earth orbit's eccentricity is just 0.0167.

Depending on the ϕ we may have hyperbolic trajectory or a closed one. Velocity of the asteroid relative to Earth may be added or subtracted from the Earth's velocity in the Solar System frame of reference and result in a velocity big enough to escape Solar System in some cases or a velocity implying closed orbit in the other. We have to remember that in our approach we analyze motion with reversed time and added velocity means that asteroid "caught up with Earth and hit it from the back" while subtracted velocity means that the opposite has happened. To find what ϕ imply the closed orbit we use energy formula. If the sum of potential and kinetic energy is negative, it means that asteroid does not have enough energy to escape to infinity and the orbit is closed. The energy is given by (1), where R_z is Earth's radius, R is mean Earth's distance from Sun, m_a is Sun's mass, m_z is Earth's mass and v is impact velocity ($25\ \frac{\mathrm{km}}{\mathrm{s}}$ in our case). On the figure 2 we see the Energy depending on ϕ angle. By solving (1) we find that closed orbits form approximately for angles in the range $[3.2305, 6.19424]$.

$$\begin{aligned} E = &-\frac{Gm_z m_a}{R_z} - \frac{Gm_s m_a}{(R + R_z\cos\phi)^2 + (R + R_z\sin\phi)^2} \\ &+\frac{m}{2}\left(\left(\sqrt{\frac{Gm_s}{R}} + v\sin\phi\right)^2 + (v\cos\phi)^2\right) \end{aligned} \tag{1}$$

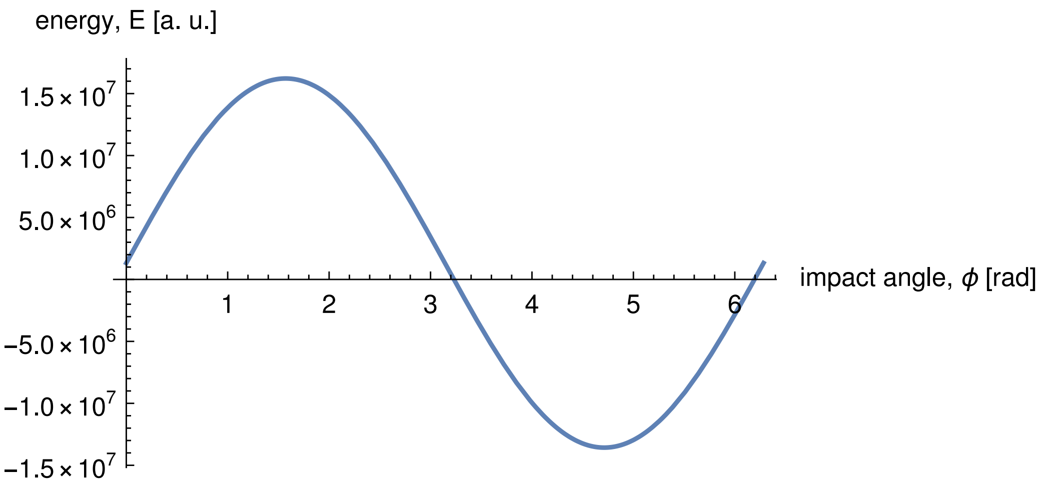

Figure 2: Dependence of energy (in arbitrary units) on impact angle – negative energy results in closed orbit.

To find possible orbits, we have simulated the motion of an asteroid starting from Earth at the angles, that result in closed orbit. Numerical simulation has been done in two phases. In phase one we calculate both the gravitational force from the Sun and Earth acting on asteroid and gravitational force from the Sun acting on Earth. At the moment when the force from the Sun acting on asteroid is more than one thousand times stronger than one acting from Earth, we move to phase two. In the phase two we assume that both the asteroid and the Earth orbit around the Sun independently and we consider only the Sun's gravitational field. This assumption in many cases is reasonable. On the figure 3 we see example of ($\phi = 4$) trajectories in phase one - time of simulation is short (below one day) and trajectories look as they would be straight. On the figure 4 we see full orbit we get for $\phi = 4$.

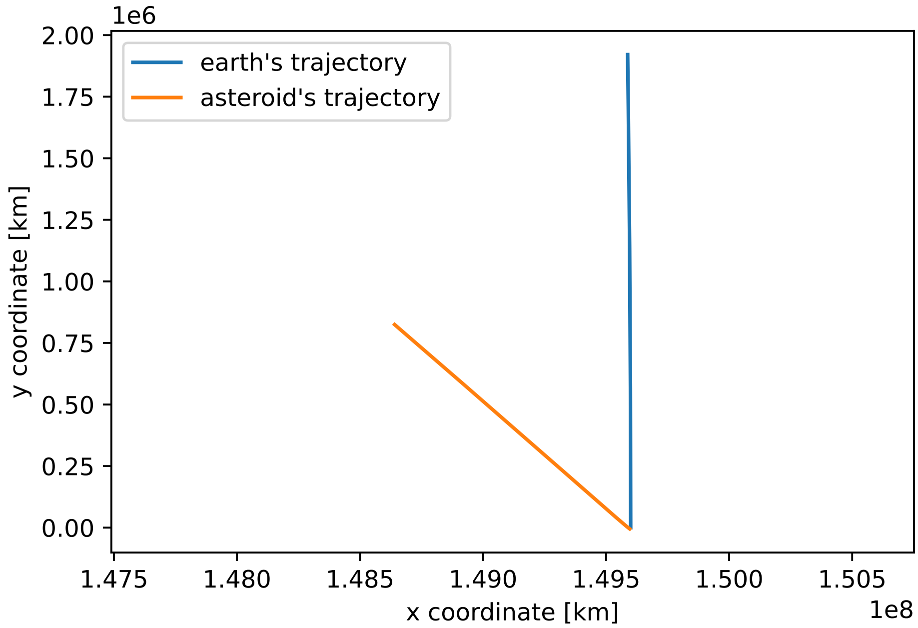

Figure 3: Trajectories in phase 1 for $\phi = 4$.

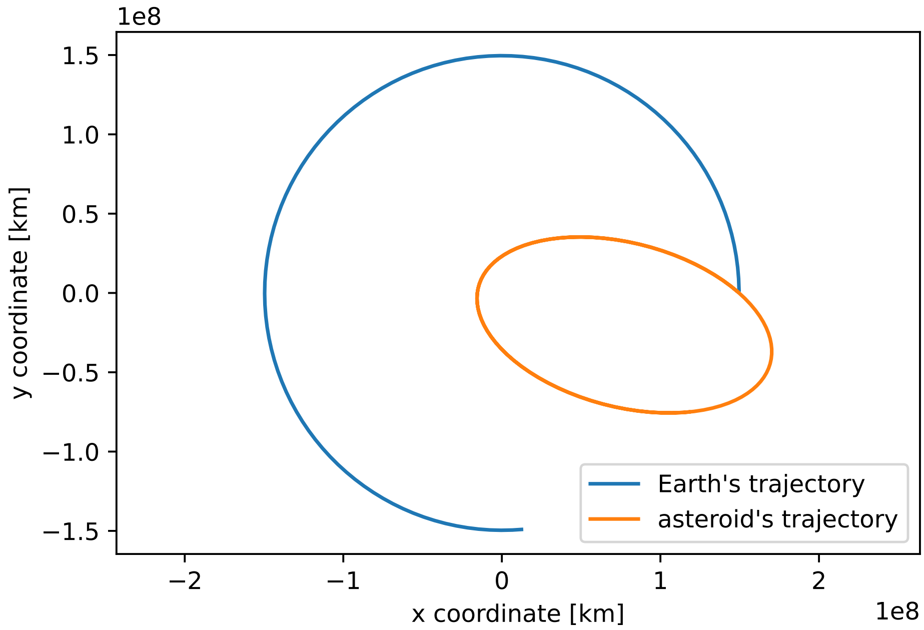

Figure 4: asteroid's orbit for $\phi = 4$.

For the purposes of its usefulness further in this work, we find dependence of the orbital period on the impact angle. This was done numerically and the results can be seen on the figure 5. As we may expect, as we approach the angles that imply a hyperbolic orbit, the period tends to infinity (we have obtained numerical results for 20 angle values between 3.35 and 6.1 radians).

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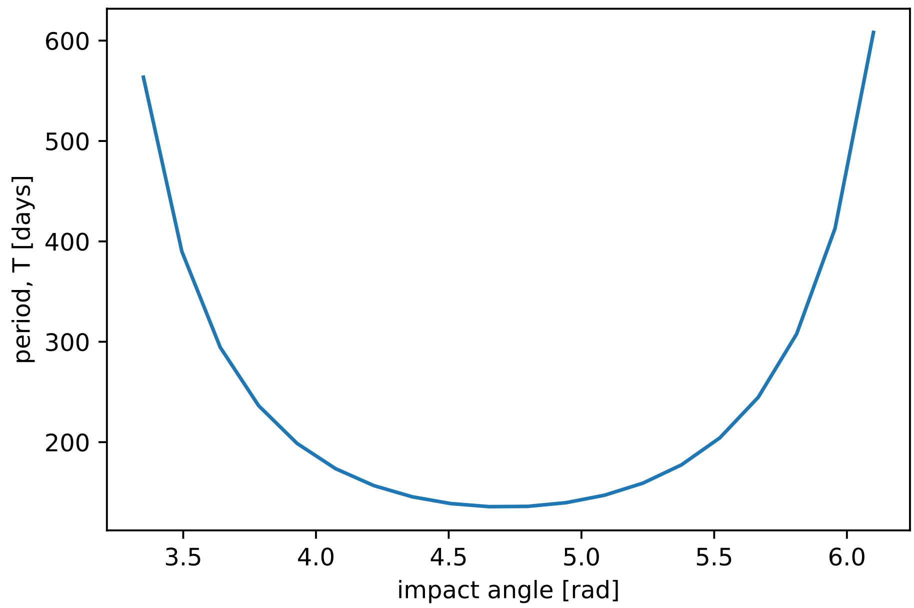

Figure 5: Dependence of period on impact angle.

To summarize - we have parameterized all the possible orbits belonging to objects mentioned in the problem statement and by a numerical simulation we have found their trajectories and orbital periods.

3.2 Model of the spacecraft and its mission

For an object potentially hazardous to Earth with diameter of 100 m, according to the comprehensive study "Defending Planet Earth" [3], two viable strategies for actively deflecting the object are a kinetic impact of a spacecraft and altering its trajectory by putting a spacecraft in proximity of the object and tracting it gravitationally. The second option requires a lot longer time before impacting the Earth to work (see regimes of applicability in the figure 6). We therefore choose our spacecraft to be designed for a kinetic impact mission and thus achieve the shortest possible time before Earth impact required for a mission success. We consider possible use of nuclear weapon to be out of scope of this problem.

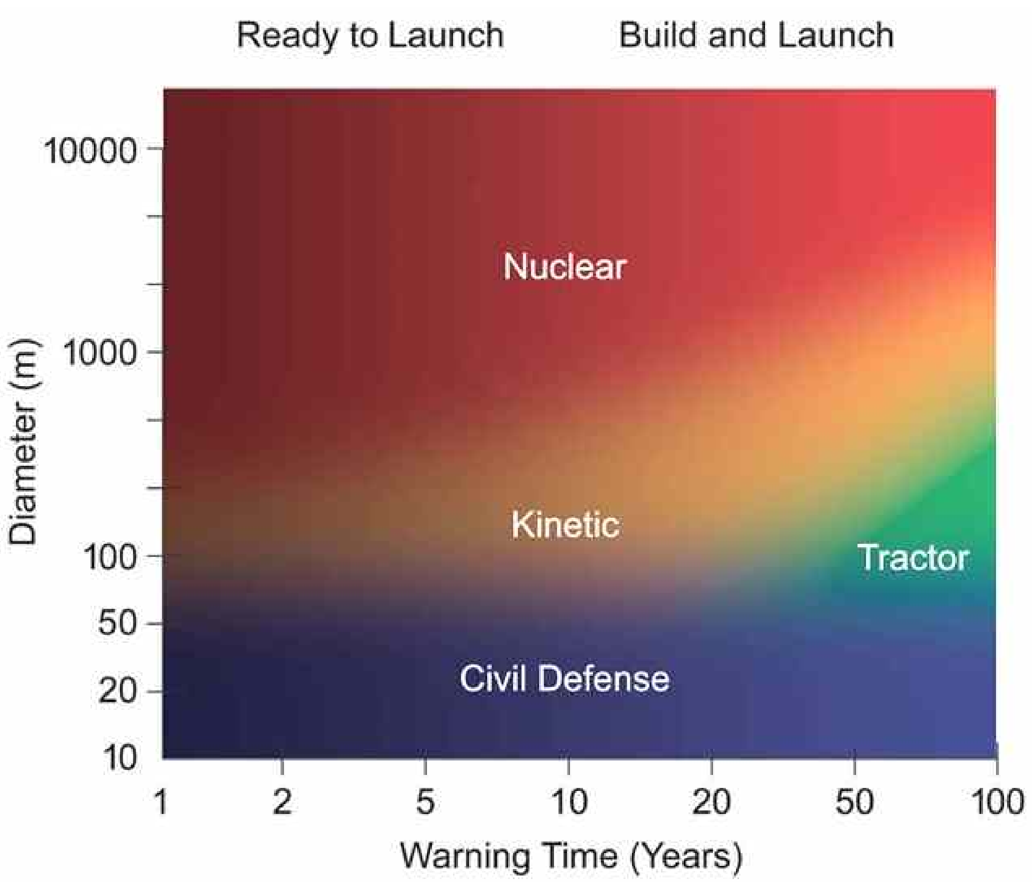

Figure 6: Range of the applicability of certain asteroid threat mitigation strategies, depending on the diameter of asteroid and time left to the impact. Source: [3]

Prime example of what are the optimal parameters of this kind of mission and spacecraft is the recent NASA Double asteroid Redirection Test. That mission's goal was very similar to what we would want to achieve in a situation described in our problem: impacting an asteroid of diameter comparable to 100 m, orbiting the Sun on an elliptical orbit nearing the Earth's one, with a maximum possible momentum transfer to the asteroid. Designing optimal trajectory and choosing the right propulsion system, constrained by the relative location of Earth and asteroid, mission time, requirement of minimum fuel loss in maneuvering and maximum kinetic energy transferred at impact is a very complex task. In this situation, designing an entire mission plan for every kind of asteroid would have to be overly simplistic and very likely introduce the difference in momentum transfer of some few orders of magnitude depending on a chosen simplifications, as well as a type of propulsion and different mission trajectories. We decide, that rather than optimizing for the best mission trajectory and designing all the propulsion usage and maneuvers, in this work we will estimate the momentum transfer from our spacecraft to the asteroid as a constant value throughout the calculations, based on the initial condition of our 20 ton spacecraft put on Low Earth Orbit and a value of momentum transfer that has been measured during a DART mission, for its smaller device. Assuming that our spacecraft, that have been put on the low earth orbit is using contemporary technology and the object we try to deflect have a typical Near Earth Object trajectory, both it and the mission trajectory can be modelled as an adequate counterpart of the DART mission, witch have had a very similar goal to ours and has been successfully carried out with the technical capabilities we have right now.

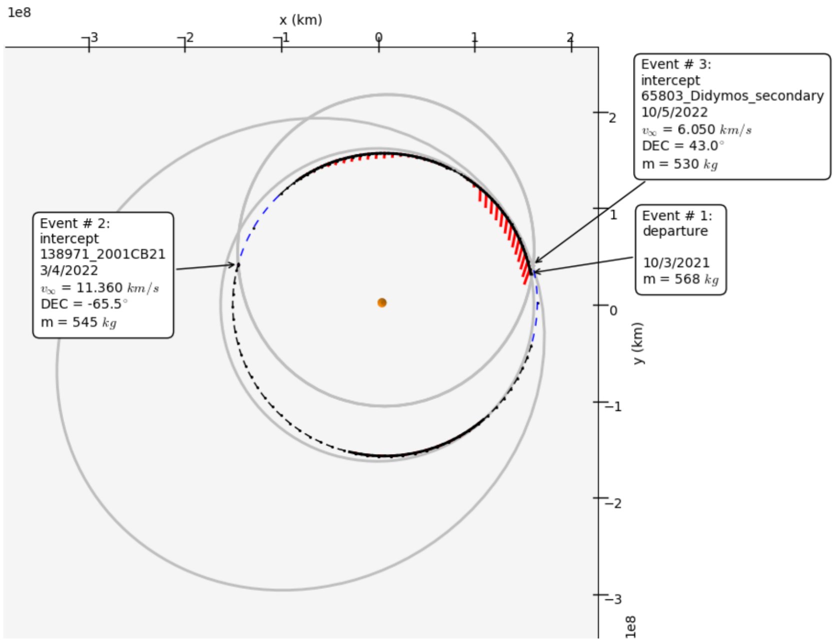

Figure 7: Optimal trajectory chosen for the DART mission. Vectors show the usage of ion thruster throughout mission. Source: [1]

Our estimation of the spacecraft momentum at impact will be taken as a product of spacecraft mass when it escapes the Earth's gravity after using some fuel and the relative velocity of spacecraft and asteroid corresponding to the results of DART mission.

3.2.1 Escaping the Earth

First, we try to estimate, how much fuel will be used up due to the escape from Earth's gravity. We start with a spacecraft on Low Earth Orbit, that is with a velocity of approximately 7.9 km/s relative to Earth. Typical Low Earth Orbit objects have the altitude of a few hundred kilometers. For example, International Space Station in its apogee has the altitude of 422 km[cite] or about 6.6% of Earth's radius. Therefore we will assume, that obtaining a total velocity equal to Earth's escape velocity (11.186 km/s) is needed for our spacecraft to escape the Earth. We use the Tsiolkovsky rocket equation:

$$\frac{m_0}{m_f} = e^{\Delta v / v_e} \tag{2}$$

Where v_e is exhaust gas velocity, equivalent to the engine's specific impulse, m_0 and m_f is the spacecraft initial and final mass, respectively and Δv is necessary difference in velocity during maneuver. We solve for m_f with given $\Delta v = v_{\text{escape}} - v_{\text{LEO}} = 11.186\ \frac{\text{km}}{\text{s}} - 7.9\ \frac{\text{km}}{\text{s}} = 3.286\ \frac{\text{km}}{\text{s}}$, $m_0 = 20 \cdot 10^3$ kg We assume that a typical liquid fuel rocket engine is used, with an effective exhaust velocity of $v_e = 3800\ \frac{\text{m}}{\text{s}}$

The result is:

$$m_f \approx 8.44 \cdot 10^3 \text{ kg} = 0.422 \cdot m_0$$

3.2.2 Impact speed

Our estimate of impact speed will be based on the measured impact velocity of the DART mission. On the September 26, 2022 the 570 kg DART Impactor spacecraft have crushed into the Dimorphos asteroid at a speed of approximately 6.1 km/s, according to the data published by the mission team [2].

This gives us a result of

$$p_{\text{est}} = 5.1463 \cdot 10^7 \text{kg} \cdot \frac{\text{m}}{\text{s}}$$

3.3 Momentum transfer during collision of the spacecraft and the asteroid

In this section we want to consider, how much momentum can the spacecraft transfer to the asteroid during a collision in real world.

Let us denote the total momentum amount carried by the spacecraft as P and let β be such parameter, that the total transfer of momentum to the asteroid is equal to βP. Let's consider three possible types of collisions (see fig.8):

1. $\beta < 1$
   This type of collision corresponds to a situation when there are some ejecta caused by the spacecraft impact carrying off momentum in the common direction with the asteroid's velocity.

2. $\beta = 1$
   This type corresponds to a perfect inelastic collision (spacecraft sticks into asteroid)

3. $\beta > 1$
   This type of collision corresponds to a situation when there are some ejecta caused by the spacecraft impact carrying off momentum in the opposite direction with the asteroid's velocity. That is the situation that helps us to affect more on the asteroid trajectory.

According to [4] the parameter β for an asteroid of diameter 100m can be about $\beta \sim 8.4$ for an impact at 10km/s. This result could be understood by the following consideration. Due to low gravity force causing by asteroid (mass of asteroid is about $M = 10^9$kg, and its radius is 50m) the escape velocity equals approximately $v_e = \sqrt{\frac{2GM}{r}} \approx 0.0052$m/s. It means that almost every object detached from asteroid reaches the escape velocity. Therefore whole powder took off by the crashing spacecraft would cause extra momentum transfer.

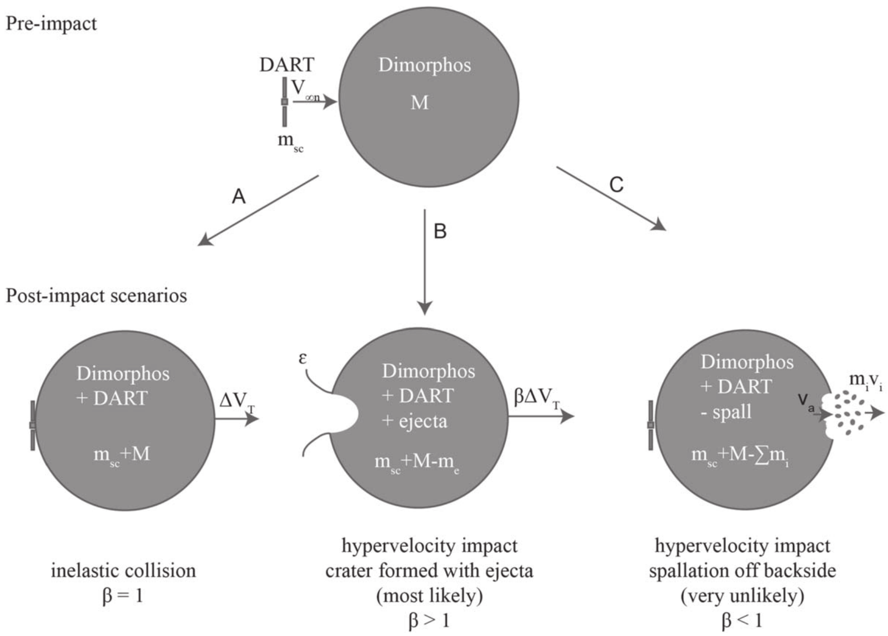

Figure 8: The picture depicts three possible types of collisions between spacecraft and asteroid. The picture comes from [1].

A requirement is to measure the momentum transfer enhancement parameter (β), which is a measure of how much additional momentum beyond that carried by the spacecraft is transferred to the asteroid in a kinetic impact. In a perfectly inelastic collision, with zero net ejecta momentum, $\beta = 1$ by definition. More generally, ejecta caused by the spacecraft impact carry off momentum, effectively giving an extra push and making $\beta > 1$ for the impact.

3.4 The optimal asteroid-spacecraft collision

One of the most important question in the problem is how the place and direction of collision between spacecraft and asteroid affects the trajectory of asteroid, and consequently the chances to hit the Earth. Therefore we put two questions: 1) how the period of a new orbit trajectory depends on the velocity direction of the spacecraft just before collision, and what is the optimal direction of it. 2) How the period of new orbit depends on the collision place on the heliocentric orbit of asteroid.
In the next part of this section we assume that the asteroid is traveling on its heliocentric trajectory $\overrightarrow{r}(t) = (x(t), y(t))$ and it is hit by the spacecraft

in moment when its position was $\overrightarrow{r_0}$. Just before collision the velocity of a spacecraft was equal to $\overrightarrow{v_0}$. Therefore the asteroid gained extra momentum coming from the spacecraft and consequently the instantaneous velocity changed from $\overrightarrow{v}$ to some $\overrightarrow{v} + \overrightarrow{\delta v}$. Using new initial conditions one can calculate new trajectory and its period $T + \delta T$, where T is the period of the trajectory before the collision.
In order to get exact results we assumed the transfer of momentum during the collision as the space craft hit the asteroid with velocity $10km/s$, density of asteroid $\rho = 2g/cm^3$ mass of spacecraft (in the moment of collision) $m = 15000kg$ and parameter $\beta = 10$. These assumptions are rather more optimistic than they could be in reality, but we assume them only to get some qualitative results.

3.4.1 The direction of the spacecraft dependence

In this subsection it is assumed that the spacecraft hits the asteroid in a fixed place $\overrightarrow{r}$. The total velocity V_s of the spacecraft is also fixed. We ask, how the change in period δT depends on the direction of the velocity of spacecraft. In this consideration we assume only trajectories on plane, so the direction of the spacecraft velocity could be characterized by only on one parameter $\theta \in (0, 2\pi)$ meaning the angle between x axis and spacecraft velocity. One can numerically check which value of θ gives the biggest change in period. In order to establish attention we made the numeric simulation for a asteroid trajectory given by parameter $\phi = 3.19$ (see fig. 9). Precisely, using numeric simulation we have calculated changed a semi-major axis change $a + \delta a$ and in consequence the change in period $\Delta T = \frac{3\sqrt{a}\delta a}{GM_\odot}$, which is the first order Taylor approximation of the third Kepler's Law $\frac{T^2}{a^3} = \frac{4\pi^2}{GM_\odot}$. The final result dependence $\Delta T(\alpha)$ is shown on the fig. 10.

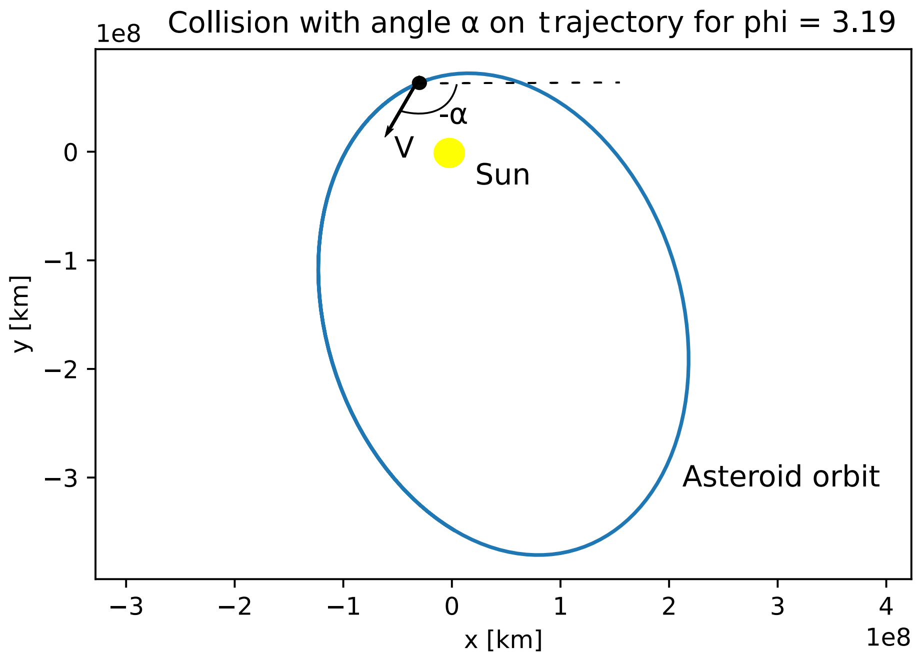

Figure 9: asteroid - spacecraft collision with the spacecraft velocity direction $\alpha \in (0, 2\pi)$ on a fixed point $\theta = 1.9$. The period change $\Delta T(\alpha)$ dependence on α is shown on the fig.10.

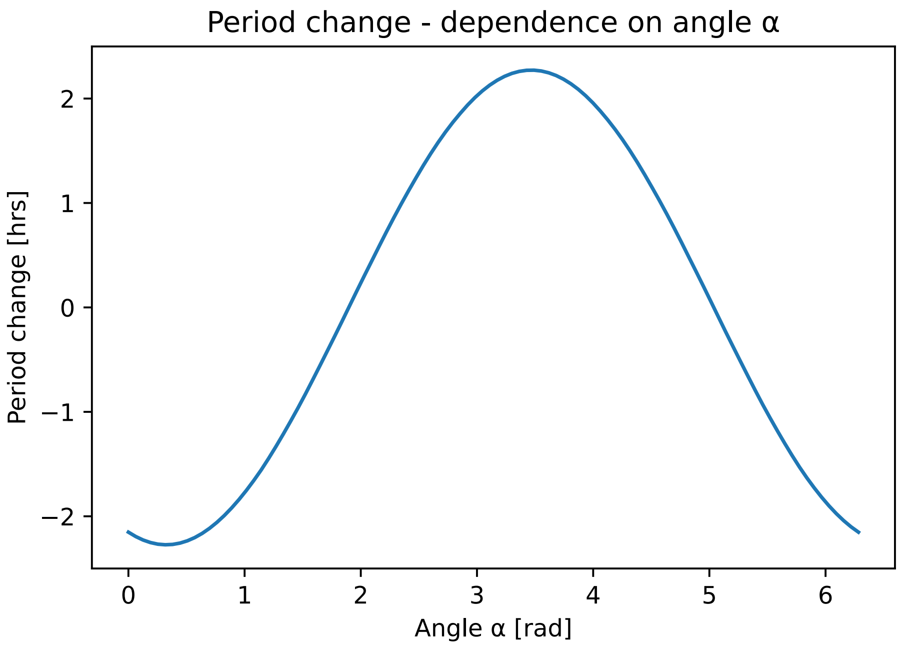

Figure 10: The resulting dependence $\Delta T(\alpha)$ for a trajectory given by $\phi = 3.19$. One could see, that vertical change of the asteroid velocity gives nearly zero effect. Therefore the best possible angle for spacecraft to hit the asteroid is parallel to the asteroid velocity.

3.4.2 The place of collision dependence

In this subsection it is assumed that the spacecraft hits the asteroid in a place $\overrightarrow{r}$ given by an angle θ in polar coordinates related to sun (see fig. 11) and the angle of spacecraft velocity is fixed (in the simulation it is parallel to the velocity vector of the asteroid with opposite direction). The total velocity V_s of the spacecraft is also fixed. We ask, how the change in period δT depends on the place around the trajectory where spacecraft hits asteroid. Analogically as before, in this consideration we assume only trajectories on plane, and we calculate the change in period from the value of semi-major axis. In order to establish attention we made the numeric simulation for a asteroid trajectory given by parameter $\phi = 3.19$ (see fig. 11).
The result of this simulation is depicted on fig.12.

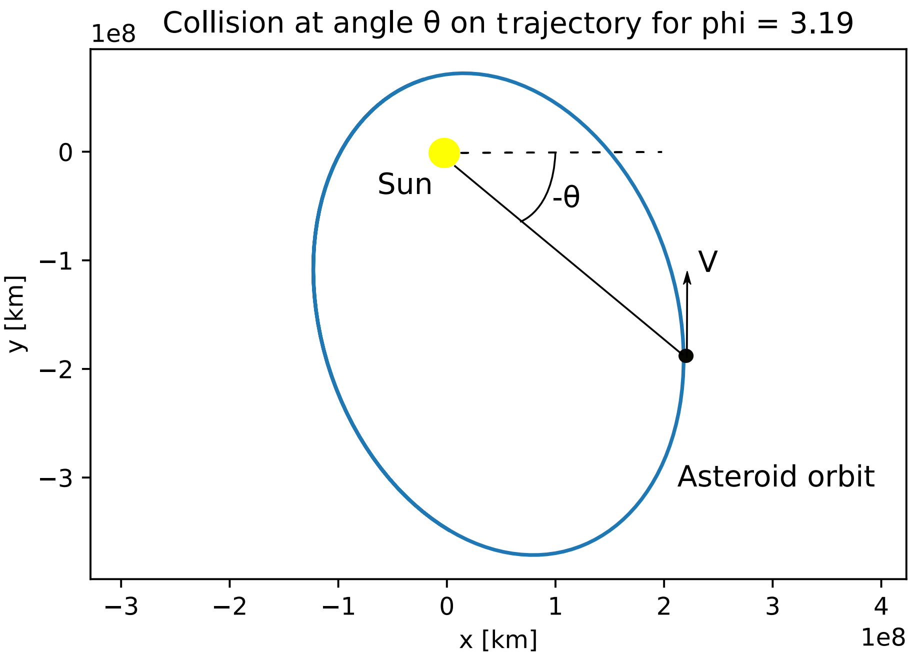

Figure 11: asteroid - spacecraft collision with the fixed spacecraft velocity direction on a point given by $\theta \in (0, 2\pi)$. The period change $\Delta T(\theta)$ dependence on θ is shown on the fig.12.

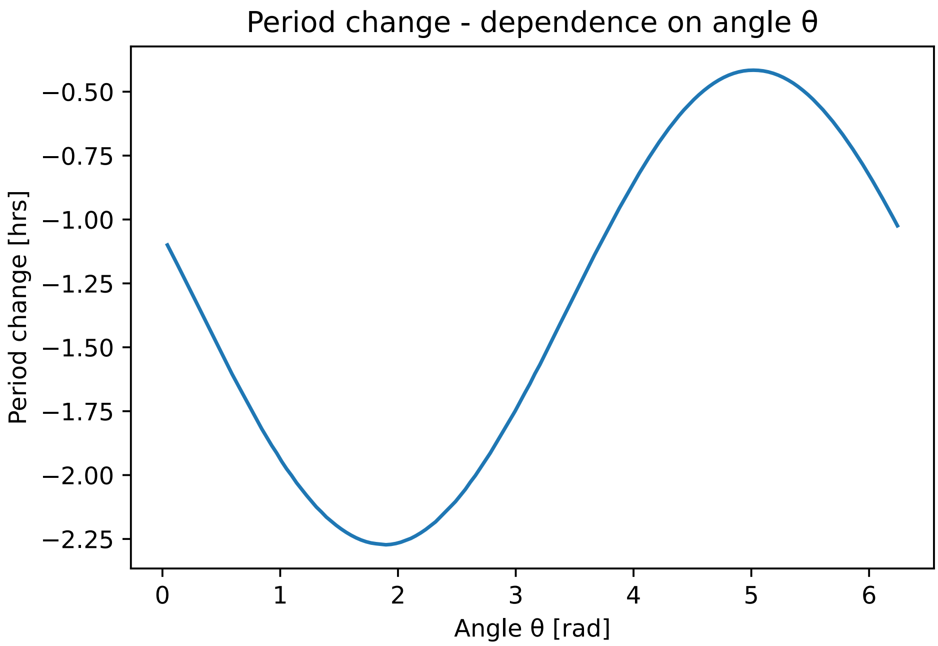

Figure 12: The resulting dependence $\Delta T(\theta)$ for a trajectory given by $\phi = 3.19$. The biggest change of period is in the angle $\theta \approx 1.9$ which correspond to the perihelion.

As we see, in the case $\phi = 3.19$ the optimal way to hit the asteroid is to do it in the perihelion and use velocity of spacecraft parallel to the velocity of asteroid. But taking another value of ϕ for closed orbits we would get the same results (just using analogical simulation). Therefore we have found the optimal way to disturb the period of asteroid and the general period dependence on place and angle of hit.

3.5 Sun - Earth - asteroid system and the conditions for thwarting an imminent collision

Our basic assumption, allowing us to simplify calculations vastly, while not departing too far away from the exact description, is that all bodies in the system are moving in one plane.

In general, when the calculations in our model concern locations not in the close proximity of Earth, its gravitational field is several orders of magnitude weaker than that of Sun and can be neglected. In the case, when we track an object that approaches Earth's proximity, however, the situation is more subtle and how we treat an effect of Earth's gravity then, will be explained

in more detail later.
The diameter of an asteroid is more than four orders of magnitude smaller than the diameter of Earth, therefore we can safely treat the asteroid as a point mass, when determining if a collision with Earth would occur. For any sensible assumption about asteroid density, also the trajectory of earth would not be altered in a measurable way by the asteroid, so we do not take that effect into the account.
For a given orbit around sun, we can have one or two x-areas, defined as the spots in Solar System, where the Earth would lie at the moment of collision, should we not alter asteroid's trajectory in any way. For a given event we set a default time and x-area, where collision is normally expected to happen and we say, that if the trajectory corrected after spacecraft hit imply no collision close to this time and area, the object has missed the Earth. We do not consider next possible events for this object, then.
The probability of an asteroid on a given orbit hitting the earth in general is so small, that after missing one unlikely opportunity for collision, that asteroid is no more likely to threat earth in the future as any other similar object.

To simplify the problem, let us consider which parameter of the trajectory is the best to use during calculations if the asteroid hits Earth.
At first, time of the asteroid-spacecraft collision is much shorter than the period of asteroid orbit, so we assume that this collision is just immediate transfer of momentum to the asteroid. Then the parameters of asteroid's trajectory change instantly. The new trajectory could miss Earth in two ways. The change of the trajectory itself and the change of the time when asteroid cross the Earth trajectory.
One could characterize the orbit trajectory of an asteroid by the period T and the semi-major axis a. They are connected with the Kepler's third law: $\frac{T^2}{a^3} = const.$. Therefore scaling a by a factor λ scale T by factor $\lambda^{\frac{3}{2}}$. After collision the change in period is greater than the change in semi-major axis. Moreover, when the semi-major axis change, it stays the same all the time, but if period changes, then the difference of time arriving becomes grater in each encirclement. In other words, when we change the period by ΔT, then after three encirclements the total time difference of crossing Earth trajectory becomes $3\Delta T$.
Taking this paragraph into consideration we use only time-difference condition of making asteroid to miss Earth.

3.6 Hit or miss - the simplest model

Having obtained the change of asteroid's period after collision we have to answer the fundamental question, that is: how it affects the collision of asteroid and Earth. We say that if $n\Delta T > \frac{R_z}{v}$, where v is Earth's velocity and n is the number of periods asteroids makes from the hit time to potential collision time, asteroid will miss the Earth. Our reasoning here is that asteroid will be in collision point $n\Delta T$ earlier comparing with the scenario in which we do not hit asteroid and do not disturb its period. So for Earth to be safe it has to be further than R_z from the collision point and from that we get our condition.

We made several approximations here. Firstly we do not consider the disturbance of asteroid's trajectory and we can make that because, as it is stated in one of previous chapters, hitting the asteroid greater disturb its period than semi axes, and period disturbance grows every period while axes disturbance stays the same, and that is why the later was neglected. Secondly we do not directly consider the attraction force between asteroid and Earth, which in first approximation may be reasonable because asteroid velocity is greater than escape velocity and even if Earth influences asteroid's trajectory asteroid will not hit Earth unless it is because their trajectories around the Sun. The later probably bigger approximation could be easily reduced by numerical simulation of the collision.

4 Result

According to section **Hit or miss - the simplest model** the minimal number of total laps must be at least $n = \frac{R_z}{v\Delta T}$. In order to get the answer in Earth's years we must rescale the result by a factor $\frac{year}{period}$. Therefore the result counted in the number of years is $y = \frac{R_z \cdot 365}{v\Delta T(\phi) \cdot T(\phi)}$., where period T is counted in days.

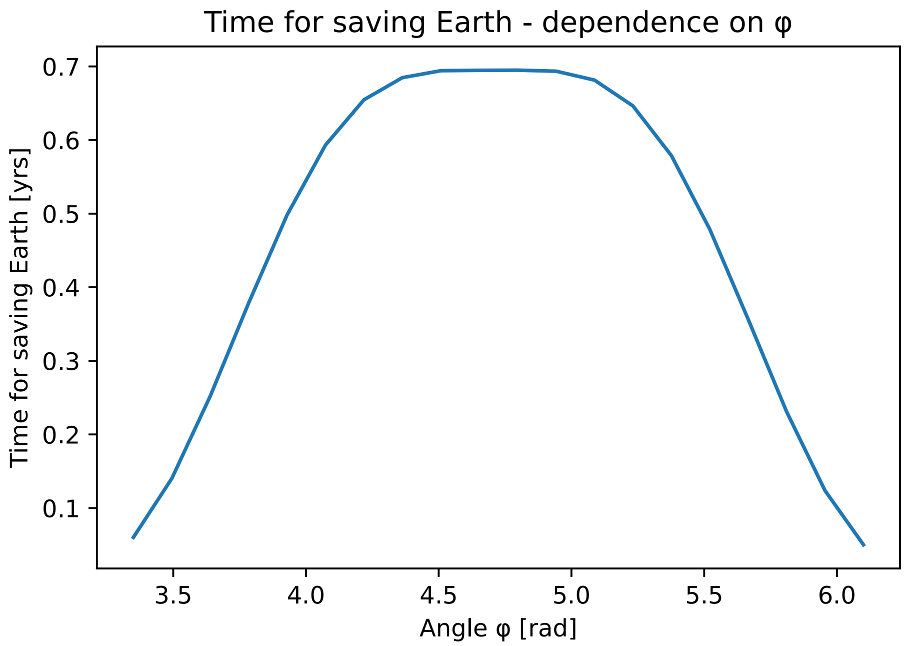

Figure 13: The minimal time amount in which spacecraft have to hit asteroid in order to avoid collision with Earth. These results come from the calculation of the period-change caused by collision with spacecraft. The approximate period of the asteroid's orbits is about $T \sim 180$ days, therefore the asteroid needs from 2 to 3 full sun encirclement to avoid collision with Earth.

5 Summary

Our attitude was to look at the problem as a description of a possible real - life threat to Earth and consider simplifications that won't lead us away from the possible realistic character of such an asteroid collision event. We have Interpreted the problem as a task of finding the approximate minimum time between a spacecraft impact to asteroid and potential asteroid impact to Earth for any possible asteroid with closed orbit around the Sun, that result in a "trajectory to directly impact the Earth at a speed of 25 km/s", understood according to the section 2.1. We assumed that the spacecraft shall only be used as a kinetic impactor. We have abstained from considering minute details of all possible spacecraft routes inside the Solar System and decided to adopt single estimate of the spacecraft momentum at impact, based to a significant extent on a recent NASA DART mission, that was very similar to the kind of spacecraft mission we would have to consider. We analyzed the very effect of the spacecraft impact on asteroid momentum, as well as a requirement for an asteroid's orbital period needed for Earth miss. We have used numerical simulations of a planetary dynamics to obtain all the results and final dependence of the minimal time before asteroid's impact to earth needed for the spacecraft, on a parameter characterizing all the relevant asteroid orbits.

References

[1] A. S. Rivkin et al., The Double asteroid Redirection Test (DART): Planetary Defense Investigations and Requirements, The Planetary Science Journal, vol. 2, no. 5. American Astronomical Society, p. 173, (2021). DOI: 10.3847/psj/ac063e.

[2] Official website of the DART impactor spacecraft; Oficcial website of the mission
https://dart.jhuapl.edu/Mission/Impactor-Spacecraft.php
https://dart.jhuapl.edu/Mission/index.php

[3] Committee to Review Near-Earth Object Surveys and Hazard Mitigation Strategies Space Studies Board, Defending Planet Earth: Near-Earth Object Surveys and Hazard Mitigation Strategies, National Research Council, p. 152, (2010). DOI: 10.17226/12842

[4] K. A. Holsapple and K. R. Housen, Momentum transfer in asteroid impacts. I. Theory and scaling, Icarus, vol. 221, no. 2. Elsevier BV, pp. 875–887, (2012). DOI: 10.1016/j.icarus.2012.09.022

[5] Loup Verlet, Computer "Experiments" on Classical Fluids. I. Thermodynamical Properties of Lennard-Jones Molecules DOI: 10.1103/PhysRev.159.98