Collision Saves the Earth

Team 460, Problem A

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Abstract

In this essay, we explore the scenario where we need to deploy a spacecraft, whose initial total mass is $20,000kg$, to collide with a coming asteroid, so as to deflect it from our Earth, and we aim to find the time t before Earth impact when our spacecraft should hit the asteroid. To do this, we first divide the whole process into three major stages to clarify our model: launching the Spacecraft, collision between the asteroid and the spacecraft, drifting of the asteroid. Before investigation into these stages, we discuss about the scales of important physical quantities to simplify our model, and to take the influence of the atmosphere into account. Afterwards, we study the three stages one by one, and acquire our result, that the spacecraft should hit the asteroid 14 years before it hits Earth, if we take the effective exhaust velocity v_r of the spacecraft as $4.4 \times 10^3 m/s$. Finally, we evaluate the result's feasibility, and point out the decisive effect of v_r on t: for $v_r = 5.0 \times 10^4 m/s$, $t = 0.67 year$.

Our model is based on physical principles and is simplified based on the scales of quantities. Therefore, our model is strong in generalizability and clarity. However, it may cost our model accuracy due to some rough estimations and presumptions.

Keywords: Rocket Equation, Collision, Conservation of Momentum, Conservation of Angular Momentum, Conservation of Energy

Contents

1 Introduction

1.1 Background

The impact of sufficiently large asteroids on the Earth would be catastrophic for mankind and other creatures because it may cause massive tsunamis or multiple firestorms. [1] A possible way to avoid asteroid impact is to launch a spacecraft and crash it into the asteroid to divert the asteroid from its course. On 26, September 2022, a probe launched by NASA intentionally crashed into Dimorphos and shortened its orbital period by about 32 minutes, which proved the feasibility of the aforementioned method. [2] Determining how much time before the impact would be needed for the spacecraft to hit the asteroid is crucial to a successful hit and is thus worth investigating.

1.2 Problem Statement

We need to launch a spacecraft, whose initial total mass is $20,000kg$, to collide with a coming asteroid at a initial velocity of $25km/s$ and with a diameter of $100m$, so as to deflect it from the Earth. We are required to find the time t before Earth impact when our spacecraft should hit the asteroid.

2 Notations

Symbols	Description
m_0	Initial mass of the spacecraft (including the fuel)
v_r	Effective exhaust velocity of the spacecraft
m_1	Mass of the spacecraft at the end of Stage 1
m	Mass of the spacecraft before the collision
u	Velocity of the spacecraft
p	Momentum of the spacecraft
p^*	Maximum momentum of the spacecraft (before the collision)
M	Mass of the asteroid
r	Radius of the asteroid
d	Diameter of the asteroid
ρ	Density of the asteroid
ρ_a	Assumed density of the asteroid
P	Momentum of the asteroid
v_0	Initial velocity of the asteroid
$v_{//}$	Component of the velocity of the asteroid in the direction to the Earth
$v_\perp$	Component of the velocity of the asteroid in the direction perpendicular to the Earth
M_E	Mass of the Earth
R	Radius of the Earth including the atmosphere
R_0	Radius of the Earth excluding the atmosphere
H	Equivalent thickness of the atmosphere
G	Gravitational constant
l	Distance between the spot of collision and the Earth
t	Time the spacecraft needs to hit the asteroid before the impact

Here the main notations are defined while their specific values will be discussed and given later.

3 General Analysis

3.1 Stages of the Entire Process

In this problem, we are required to launch the spacecraft from its original orbit to the spot of collision, design its collision with the asteroid, and then monitor the asteroid's trajectory so that it will not collide onto the Earth. Thus, the physical process can be divided into three major stages:

launching the spacecraft to the collision spot; collision between the asteroid and the spacecraft; drifting of the asteroid to the Earth.

1. In the first stage, we need to launch the spacecraft from its low-earth orbit to the collision spot. Since spacecrafts need to eject burned fuel products to accelerate, this process will affect the spacecraft's mass when it reaches the spot.

2. In the second stage, we need to adjust the spacecraft to a certain direction and accelerate it to a certain speed, so that it may have the maximum impact on the asteroid's future trajectory. We also need to calculate the asteroid's final velocity.

3. In the third stage, we need to analyze the asteroid's movement after the collision. In order to calculate the time the collision needs to happen before the impact, we study the case where the asteroid barely misses the Earth.

There are also possible minor stages, such as the case where the asteroid passes through the Earth's atmosphere. They will be qualitatively discussed in the next subsection on the basis of scales of important quantities.

3.2 Scales of Important Quantities

It is also important to consider the scales of important quantities, so that we may qualitatively decide on the nature of some physical processes, and simplify our model or calculation in some steps. Below are some important quantities to consider:

3.2.1 The Spacecraft's Effective Exhaust Velocity v_r

The spacecraft's effective exhaust velocity v_r is defined by the relative velocity to the spacecraft at which the burned fuel products are ejected. It is crucial to further calculations of the spacecraft's acceleration process. This parameter differs from type to type, and we choose $v_r = 4.4 \times 10^3 m/s$, which is the effective exhaust velocity of a typical liquid-fuel rocket, as our reference value. [6]

3.2.2 Scale of the Asteroid and its Momentum Vs. the Spacecraft

There are many categories of asteroids, and their densities vary accordingly. As shown in Figure 1, we can see that the density of most asteroids falls in the range of $1.0 \sim 5.0 g/cm^3$. [8] Due to this lack of information, we assume the density of the asteroid $\rho = \rho_a = 2.0 \times 10^3 kg/m^3$ to calculate the final values as a reference, and also to compare the scales of some quantities.

Here, we have the mass of asteroid $M = \frac{1}{6}\pi d^3 \rho = 5.236 \times 10^5 \rho$. Assume $\rho = \rho_a$, $M = 1.0 \times 10^9 kg$. Since the initial mass of the spacecraft $m_0 = 20000 kg$, $\frac{m_0}{M} \sim 10^{-5}$, the spacecraft's mass at any stages is insignificant compared to that of the asteroid.

The momentum of the asteroid is given by $P = Mv_0 = 2.6 \times 10^{13}$; the maximum momentum of the spacecraft (without initial velocity) is given by $p = \frac{1}{e} v_r m_0 = 3.2 \times 10^7 kg \cdot m/s$ (this equation will be proved in the second stage). Thus, $\frac{p}{P} \sim 10^{-6}$, which implies that the spacecraft's momentum is also insignificant compared to the asteroid.

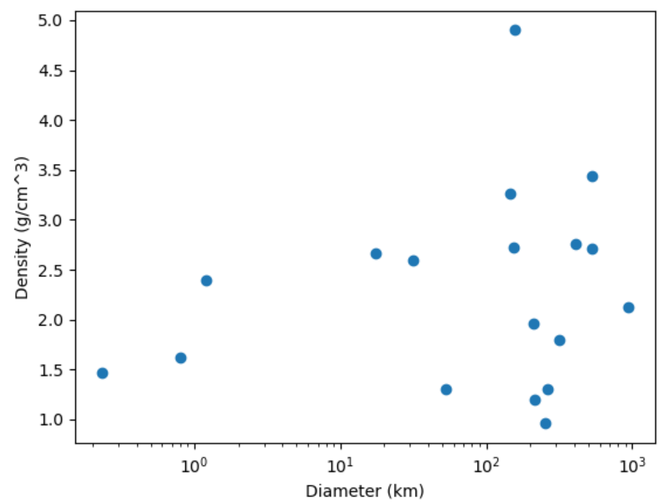

Figure 1: Diameter v.s. Density of Asteroids

Therefore, we can deduce that it is impossible for the spacecraft to collide with the asteroid head-on and push it back; instead, the spacecraft needs to give the asteroid a side-push and make its trajectory deviate. This will be quantitatively proved in the second stage.

3.2.3 Scale of the Atmosphere and its Effect

The atmosphere contains particles, which, when the asteroid passes through, will exert a drag force on it in the opposite direction to its velocity. [3] This force will decrease the asteroid's velocity and thus decrease its angular momentum, which may cause the asteroid to fall to the ground when it would otherwise barely passes through the Earth. Therefore, we should add the height of the atmosphere H to the Earth's radius R_0, and keep the asteroid out of the final sum $R = R_0 + H$. We will briefly discuss about what value of H we should take.

The Earth's atmosphere can extend to $10,000km$ if the Exosphere is included, which is even larger than the Earth's radius $R_0 = 6378.1km$. However, both the Exosphere and the Thermosphere contain too less particles and won't interfere with the movement of the asteroid. Thus, we only need to consider the layers below, which only extends to $80km$'s height from the ground.

In addition, the atmosphere's density decreases exponentially with the increase of altitude: [4,5]

$$\rho_{air}(h) = \rho_{air}(0)e^{\frac{h}{H_n}} \tag{1}$$

where h is the altitude, and H_n, the scale height, or the increase in altitude for which the atmospheric pressure decreases by a factor of e, is equal to $10.4km$ for air. Also, as for the air drag,

$$F_D = \frac{1}{2}\rho_{air}v^2C_DA \tag{2}$$

where F_D is the drag force, v is the asteroid's speed with respect to the air, C_D is a constant($C_D = 0.47$ for spheres), and A is the reference area ($A = \pi r^2$ for spheres, with r being the radius), from which we can see $F_D \propto \rho$.

Thus, when calculating the work the air drag force does on the asteroid,

$$W_D = -\int_0^{+\infty} F_D dh \propto \int_0^{+\infty} \rho_{air}(h)dh$$

$$\int_0^{+\infty} \rho_{air}(h)dh = H_n$$

Therefore, we can use H_n as the equivalent height of the atmosphere.

Since this estimation is rather rough, we only retain one significant figure of H_n, so $R = R_0 + H_n = 6.39 \times 10^6 m$. We will use R instead of R_0 as the Earth's radius in the calculation below.

4 First Stage: Launching the Spacecraft

4.1 Introduction, Presumptions and Observations

First, we look into the stage when the spacecraft travels from the Earth to the spot of collision. We want to study in this stage how much of its initial mass m_0 is lost. In order to simplify our calculation, we design this process such that the spacecraft is accelerated for a short time from its initial speed u_0 to a speed u_1, and then drift to the spot of collision, and when it reaches there, all its speed is lost; the velocity always points towards the collision spot. We further assume that:

1. The acceleration process is short enough, so that we can ignore the gravitational force of the Earth in this process. (This will be compensated when considering the change of gravitational energy, where we choose the starting point as the original orbit of the spacecraft.)
2. The spot of collision is far enough, so that the spacecraft's gravitational energy equals to zero (choose the points infinitely far away for the Earth as zero potential point) when the spacecraft gets there. (This will be confirmed in the third stage.)

4.2 Calculation of m_1

Since the spacecraft is initially in low Earth orbit, its gravity acts as the centripetal force.

$$\frac{u_0^2}{R_0 + h} = \frac{GM_E}{(R_0 + h)^2} \tag{3}$$

where u_0 is the initial velocity of the spacecraft, h is the distance from the low Earth orbit to the Earth's surface, G is the gravitational constant, R_0 is the radius of the Earth without the atmosphere, M_E is the mass of the Earth, and H is the equivalent thickness of the atmosphere. Here we take $h = 2,000km$. We thus get $u_0 = 6.793 \times 10^3 m/s$.

We then consider the acceleration process. The rocket's final speed is given by the rocket equation: [7]

$$u(m) = u_0 + v_r ln\frac{m_0}{m} \quad (u(m) > u_0) \tag{4}$$

where u is the velocity of the spacecraft, m_0 is the initial mass of the spacecraft ($m_0 = 20,000kg$), and v_r is the effective exhaust velocity of the spacecraft.

In order that the rocket can reach the collision spot, by the conservation of energy, we derive that:

$$\frac{1}{2}mu^2 - G\frac{M_E m}{R_0 + h} = 0 \tag{5}$$

Combining the two equations, we get $u = 9.607 \times 10^3 m/s$, and $m = 1.055 \times 10^4 kg$, which is the value of m_1, i.e., the mass of the spacecraft at the end of the first stage.

5 Second Stage: Collision Between the Asteroid and the Spacecraft

5.1 Introduction, Presumptions and Observations

In this stage, we presume that the spacecraft has already arrived at the point where it is to collide with the asteroid, with its remaining mass m_1, and that it can quickly speed up to achieve its maximum momentum and adjust its direction to crush onto the asteriod as shown in Figure 5.1. We want to first determine the maximum momentum p^* it can achieve, and then study its collision with the asteroid and find the asteroid's velocity after the collision.

We further make the following presumptions and observations based on the scale of quantities:

1. Both the spacecraft and the asteroid can be modeled as mass points. The diameter of the asteroid d is $100m$; the dimensions of a standard spacecraft are within the range of $1 \sim 10m$. Thus, the spacecraft can be safely modeled as a mass point; the validity of the second statement will be proved in the calculation.

2. $m_1 << M$, $m << M$, and $p << P$, as is shown in Section 3.2.2. $\frac{p}{P}$ is even smaller because the mass of the spacecraft has decreased.

3. The collision is perfectly inelastic. The spacecraft will not stand such a collision and will thus explode. We assume that the debris will fly off in random directions with respect to the asteroid, so the center of mass of the whole spacecraft is with the same velocity as the asteroid.

4. The debris's velocity only remains the same as that of the asteroid after the collision for a short time. Therefore, the spacecraft's mass needs to be accounted only in this stage, but not in the next stage.

5.2 Calculations of m, p^*, α, $v_\perp$ and $v_{//}$

We want to first study the maximum momentum the spacecraft can achieve given its initial mass m_1, so that it can give the maximum impact to the asteroid. Since the spacecraft loses all of its velocity when it reaches the spot, we have

$$u(m) = v_r \ln \frac{m_1}{m}$$

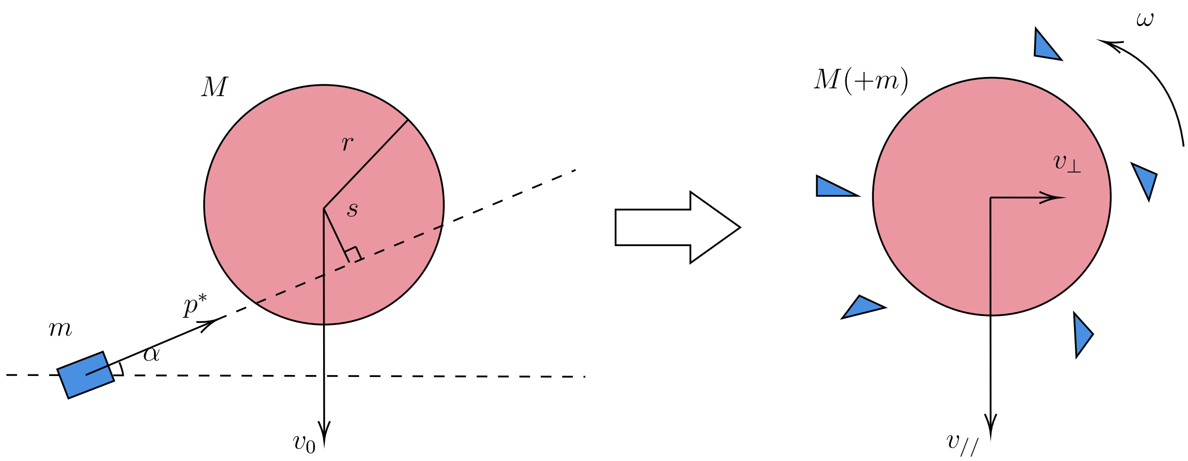

Figure 2: Collision Between the Asteroid and the Spacecraft

$$p(m) = v_r m \ln \frac{m_1}{m}$$

where u is the velocity of the spacecraft, and p is of course the momentum of it.

To maximize $p(m)$, we solve $\frac{dp}{dm} = 0$ and get the remaining mass

$$m = \frac{m_1}{e} = 3.881 \times 10^3 kg$$

and the maximum momentum of the spacecraft

$$p^* = \frac{v_r m_1}{e} = 1.708 \times 10^7 kg \cdot m/s$$

This also proves the equation we used in Section 3.2.2.

During the collision, both the momentum and the angular momentum of the system are conserved. As for conservation of momentum, we have:

$$M v_0 - p^* \sin \alpha = (M + m) v_{//} \tag{6}$$

$$p^* \cos \alpha = (M + m) v_\perp \tag{7}$$

where M is the mass of the asteroid, and denote by $v_{//}$ and $v_\perp$ respectively the component of the velocity of the asteroid in the direction to the Earth, and perpendicular to the Earth. Then, we get

$$v_{//} = \frac{M v_0 - p^* \sin \alpha}{(M + m)}, \quad v_\perp = \frac{p^* \cos \alpha}{M}$$

Since $p^* << P$, $m << M$, we can simplify the Equation (6) to $v_{//} = v_0$ within 10^{-3} of relative error. This means we shouldn't waste any of the spacecraft's velocity on pushing back the asteroid, because the effect wound be insignificant (this statement will be further confirmed in the next section). Thus, we should take $\alpha = 0$, and we can write the Equation (7) as $v_\perp = \frac{p^*}{M}$. Thus, we have $v_{//} = 2.5 \times 10^4 m/s$ and $v_\perp = 0.016 \times m/s$. We can observe that

$$\frac{v_\perp}{v_{//}} = \frac{p^*}{P} \sim 10^{-6}$$

Next, using conservation of angular momentum, we have:

$$p^* s = (I + mr^2)\omega \tag{8}$$

where $I = \frac{2}{5}Mr^2$ because of the asteroid's spherical shape.

Thus, $\omega = \frac{p^* s}{I}$, and as for the kinetic energy of the asteroid after collision

$$E_k = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_\perp^2 + \frac{1}{2}Mv_{//}^2 = \frac{5}{4}\frac{p^{*2}s^2}{Mr^2} + \frac{1}{2}\frac{p^{*2}}{M} + \frac{1}{2}Mv_0^2 \tag{9}$$

The term of rotational kinetic energy $\frac{5}{4}\frac{p^{*2}s^2}{Mr^2}$ is insignificant compared to E_k ($\sim 10^{-12}$) even when $s = r$ and it reaches its maximum; in addition, we will see in the next stage that no force will interfere with the asteroid's rotation, so this proportion of energy will not make any difference. Therefore, modelling the asteroid as a mass point is also reasonable.

6 Third Stage: Drifting of the Asteroid

6.1 Introduction, Presumptions and Observations

At the beginning of this stage, the asteroid retains its velocity $v_{//} = v_0$ towards the Earth and acquires a velocity $v_\perp = p^*/M$. We want to study the case where it barely passes by the Earth without hitting it, and then find out the minimum distance l where the spacecraft has to collide with the asteroid, and thus the time t the collision needs to happen before the impact. The diagram is shown in Figure 3.

We here observe and presume that:

1. The final velocity v_f should be tangential to the Earth's surface.

2. During this drifting stage, the asteroid is only affected by the Earth's gravitational force. This presumption will be fallible if d is so large that the asteroid is likely to be affected by other celestial bodies' gravitational field, and in that case, predicting its trajectory will be extremely difficult. We will come back to this presumption after we have calculated the value of l and provide justification.

6.2 Calculations of v_f, l and t

During this stage, because the asteroid is only subject to a central, conservative force, the gravitational force of the Earth, its angular momentum with respect to the Earth's center and the energy of the asteroid-Earth system is conserved. Choosing the points infinitely far away for the Earth as zero potential point, we have:

$$v_\perp l = v_f R \tag{10}$$

$$\frac{1}{2}v_{//}^2 - G\frac{M_E}{l} = \frac{1}{2}v_f^2 - G\frac{M_E}{R} \tag{11}$$

we only retain the term $\frac{1}{2}v_{//}^2$ to represent the asteroid's kinetic energy after collision, because other two terms are too insignificant, as discussed in Section 5.2.

We first look at the second equation. Since $\frac{1}{2}v_{//}^2 = 3.13 \times 10^8 J/kg$, $\frac{GM_E}{R} = 6.23 \times 10^7 J/kg$, and $-\frac{GM_E}{R} < -\frac{GM_E}{l} < 0$, we can deduce that $25km/s < v_f < 27.4km/s$, or $v_0 < v_f < 1.1v_0$. Turning to the first equation, we acquire that $l = \frac{v_f}{v_\perp}R$, where $\frac{v_\perp}{v_{//}} \sim 10^{-7}$. Therefore, $\frac{l}{R} \sim 10^7$, and we can safely ignore the term $-G\frac{M_E}{l}$. This also confirms our presumption in Section 4.1.

Thus, we can calculate v_f from the second equation: $v_f = 2.74 \times 10^4 m/s$. From the first equation, then, we derive $l = 1.1 \times 10^{13} m$ (only 2 significant figures are retained because ρ, thus M, thus $v_\perp$ only has 2 significant figures).

Therefore, t is given by $t = \frac{l}{v_{//}} = 4.3 \times 10^8 s = 5.0 \times 10^4 day = 14 year$.

7 Evaluation

7.1 Evaluation of the Result

The result shows that the collision needs to happen about 14 years before the impact. This is a very large number compared with NASA's plan, where the probe was launched only 10 months, or 0.83 years before the collision. [2] Such a long drifting time may make the spacecraft susceptible to attrition and extra influences like other passing celestial bodies. In addition, $l = 1.1 \times 10^{13} m >> 1AU = 1.5 \times 10^{11} m$, which may be too far a distance. Thus, this result is not satisfying.

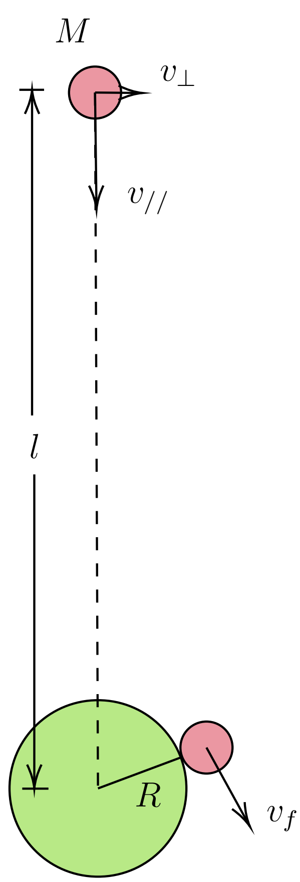

Figure 3: Drifting of the Asteroid

Such a long time mainly comes from the relatively huge mass of the asteroid, and the small value of $v_r = 4.4 \times 10^3 m/s$. If we increase the value of v_r to $5.0 \times 10^4 m/s$, which can be generated by the Hall-effect thruster, we get $m_1 = 1.891 \times 10^4 kg$, $p^* = 3.477 \times 10^8 kg \cdot m/s$, $v_\perp = 0.332 m/s$, $l = 5.3 \times 10^1 1m$, and can reduce t to $2.4 \times 10^2 day$ or $0.67 year$. This result is satisfying, and shows the decisive influence of v_r on our result. Thus, it is essential that technology that can further increase v_r is developed.

7.2 Evaluation of the Model

Our model is based on physical principles instead of data analysis; meanwhile, based on the scales of quantities, the model and some calculations are simplified. Thus, our model enjoys the following strength:

7.3 Strength

- Since the model is based on physical principles, it is easy to generalize our model to similar cases.
- Analysis of the scales of quantities greatly simplifies our model and makes it more understandable. Moreover, it helps to clarify the main stages of the model.

Our model also has the following weaknesses:

7.4 Weakness

- Some parameters, such as the asteroid's density ρ, has few significant figures due to lack of information. This may decrease the accuracy of our model.
- Some estimations we have made, such as taking the equivalent thickness of the atmosphere H as H_n, may be too rough, and can decrease our accuracy. (However, here, since the atmosphere's possible equivalent thickness is much smaller than the Earth's radius M_E, the inaccuracy is well-controlled.)

8 Conclusion

In this essay, we explore the scenario where we need to deploy a spacecraft to collide with a coming asteroid, so as to deflect it from our Earth. We first divide this process into three major stages to clarify our model, and discuss about the scales of important physical quantities to simplify it. Then, we study the three stages one by one, and acquire our result $t = 14 year$ for the parameter $v_r = 4.4 \times 10^3 m/s$. Finally, we evaluate the result's feasibility, and point out the decisive effect of v_r on t.

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