Research on Rapid Aiming of Cannon

Team 127 Problem B

November 10, 2025

Abstract

For artillery officers, how to rapidly target and engage objectives is a crucial matter.This paper aims to explore the ballistic calculation of muzzle-loading smoothbore cannons before the 19th century. Focusing on spherical projectiles of 5 kilograms mass and 11-cm diameter , it investigates the relationship between required initial velocity and firing angle for target hits under specific coordinate and environmental conditions. Models were successively established for four scenarios: ideal conditions, incorporating air resistance, varying target positions, and wind speed effects. The RK45 numerical method was employed to solve the equations of motion, determining the requisite initial velocity and firing angle. To ensure sufficient impact energy,firing should occur at a low angle. The relationship between firing angle and initial velocity when striking the reference point is $\theta_L = 0.0001{v_0}^2 - 0.1145v_0 + 29.312$. Horizontal distance causes an angular deviation of 0.2 degrees per 50 meters. Altitude causes an angular deviation of 1.1° to 1.2° per 25 meters. Wind speed causes an angular deviation of 0.01 degrees per 1 meter per second.Finally, we developed a set of simple quick-aiming mnemonics for artillery officers.

Keywords: artillery shell, initial velocity, firing angle, numerical method

Contents

1 Introduction and Problem Statement

As a time-honored weapon system, cannon has long played a decisive role in military operations, yet achieving precise strikes under real combat conditions remains a persistent challenge. Battlefields are dynamically complex: rapid shifts in enemy positions and adverse weather such as strong winds can all disrupt projectile trajectories. Artillery units often need to deploy hastily with limited aiming time, making rapid target acquisition and engagement crucial for artillery officers.

Figure 1: Civil War cannon and cannonball stack at Stones River National Battlefield, from the public domain resources of the U.S. National Park Service .[1]

Our objective is to determine the muzzle velocity and firing angle above the horizontal required to hit a target at a horizontal distance of 1,200 meters with an 11-cm diameter projectile weighing 5 kg, given a muzzle velocity of up to 450 m/s. The analysis is then extended to targets at horizontal distances between 1,000 and 1,500 meters, under varying altitudes and wind conditions. The resulting method provides artillery officers with a rapid, manual estimation technique for engaging targets effectively in the field.

2 Assumptions and Analysis

2.1 Assumptions

1. The projectile is a regular, rough sphere, and the air resistance it encounters acts uniformly on the projectile.

2.Throughout its trajectory, the projectile experiences a gravitational force that is assumed to be constant.

3.When hitting a high target, the projectile's dimensions are negligible, such that it can be considered a point mass.

4.During flight, the projectile is affected only by gravity, air resistance, and wind resistance.

5. In modeling the effect of wind on a projectile's trajectory, both the wind velocity and direction are held constant.

2.2 Analysis

Whether a projectile can hit a predetermined target is influenced by multiple factors. Below we conduct a comprehensive analysis:

1. Muzzle Value Factors

A specific muzzle velocity must be fired at a specific firing angle to hit the target. Due to trajectory diversity, the solution to this problem is not unique.

2. Air Resistance Impact

Since the aerodynamic drag force depends on air density, and air density varies with altitude, the change in drag with altitude must be considered.[2]

3. Target Size

Historically, this cannon saw extensive use against city walls in siege warfare.[3][4] The projectile is negligible in size compared to the target. To ensure effective impact, the projectile's terminal velocity should not be excessively low.

4. Wind Effects

Headwinds create drag alongside air resistance, slowing the projectile. Crosswinds cause deflection in the final impact point. To maintain accuracy, the muzzle should be appropriately deflected.

3 Notations

Variables	Explanations	Value
g	Gravity Acceleration	$9.81m/s^2$
v_0	The muzzle velocity	
a	The acceleration of a projectile	
h	The height of cannon	
R	Distance between the target and the cannon	
H	The altitude of the target	
θ_0	Firing angle above the horizontal	
θ_H	High Elevation	
v_H	High-angle muzzle velocity	
t_H	High-Angle Velocity Time-of-Flight	
θ_L	Low Elevation	
v_L	Low-angle muzzle velocity	
t_L	Time-of-Flight at Low Angle Velocity	
C_d	Drag Coefficient	

ρ_0	Air resistance at 500 meters altitude	$1.125\ kg\ /\ m^3$
r	The radius of the projectile	
t	Time of Flight	
v_{cw}	The Crosswind Velocity	
v_{hw}	The Headwind Velocity	
θ_{hw}	Angle of the headwind	
H_0	The height of the target	
H_b	Characteristic height	$8500\ m$
F_d	Air Resistance	

Table 1： Notations

4 Modelling

The projectile is fired with a muzzle velocity of v_0 and a firing angle above the horizontal of θ_0.The target is assumed to be a city wall with a height of 10 meters.

4.1 Simple Case

Ignoring air resistance, the projectile is subject only to gravity, which constitutes a simple case of projectile motion. Horizontally, the object's velocity remains constant, while vertically it experiences acceleration downward at a magnitude of g . By resolving the velocity into its orthogonal components, we obtain

$$\begin{cases} v_x &= v_0 \cos\theta_0 \\ v_y &= v_0 \sin\theta_0 - gt \end{cases} \tag{1}$$

In the vertical direction, we can consider the projectile to be undergoing a vertical projectile motion. The total time spent in the air is $\frac{2v_0 \sin\theta_0}{g}$,and the distance traveled horizontally is $R = v_0 t \cos\theta_0$.After simplification, the final expression can be obtained,that is：

$$R = \frac{v_0^2 \sin 2\theta_0}{g} \tag{2}$$

Through this expression,and $R = 1200\text{m}$,we obtain the relationship between the muzzle velocity and the firing angle in a simplified scenario.

4.2 Considering Air Resistance

When an object moves through the air, it encounters air resistance that forces it to slow down. Consequently, the projectile's trajectory is no longer a simple parabola. In the following model, we will consider the cannon as being simultaneously affected by both gravity and air resistance.

The general expression for air resistance is $F_d = \frac{1}{2} c_d \rho A v^2$, where c_d is the drag coefficient.[5] Assuming the projectile is a regular sphere, c_d is taken as 0.47;[6] ρ is the air density at that altitude; A is the frontal area projected perpendicular to the direction of relative motion, which for a projectile is $A = \pi r^2$; and$v$ is the magnitude of the relative velocity.Furthermore, considering that air density varies with altitude, we take $\rho = \rho_0 e^{-\frac{h}{H_b}}$, where ρ_0 is the air density at sea level,h represents the altitude of the projectile above a reference point , and H_b is the reference altitude.[7] Finally, the expression for the magnitude of air resistance is obtained as

$$F_d = \frac{1}{2} C_d \rho_0 e^{-\frac{h}{H_b}} A v^2 \tag{3}$$

During flight, the projectile consistently satisfies $v^2 = v_x^2 + v_y^2$,where v denotes the projectile's velocity at a given instant,v_x denotes the horizontal component of the velocity at that instant, and v_y denotes the vertical component of the velocity at that instant. Therefore, we can also resolve the air resistance into its orthogonal components to obtain：

$$\begin{cases} F_{d\perp} & = \frac{1}{2} C_d \rho_0 e^{-\frac{h}{H_b}} A v v_y v_0 \cos\theta_0 \\ F_{d\parallel} & = \frac{1}{2} C_d \rho_0 e^{-\frac{h}{H_b}} A v v_x v_0 \sin\theta_0 - gt \end{cases} \tag{4}$$

Air resistance continuously exerts a decelerating force on the projectile.

Taking into account both gravity and air resistance, the force analysis of the the projectile is shown in the figure:

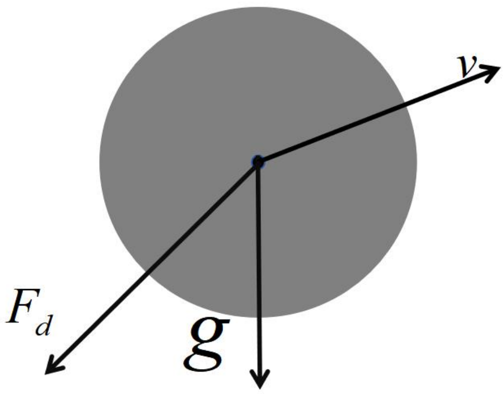

Figure 2. Force Analysis for projectile

We obtain the magnitude of acceleration in two directions, respectively:

$$\begin{cases} a_x & = -\frac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v_x}{2m} \\ a_y & = -\left(g + \frac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v_y}{2m}\right) \end{cases} \tag{5}$$

The resulting equations of motion in two directions are:

Horizontal direction:

$$\begin{cases} v_x & = v_0 \cos\theta_0 - \dfrac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v_x}{2m} t \\ R & = \int v_x \, dt \end{cases} \tag{6}$$

Vertical direction:

$$\begin{cases} v_y & = v_0 \sin\theta_0 - \left(g + \dfrac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v_y}{2m} \right) t \\ h & = \int v_y \, dt \end{cases} \tag{7}$$

This is a system of partial differential equations with exponential and quadratic terms, making analytical solutions challenging. We employ numerical methods to solve it, defining an error function to obtain results within a specified range.

Considering the target size, we set it to 10 meters in height, with the projectile being negligible relative to the target's dimensions. Thus, if the projectile trajectory intersects with the $\mathrm{R} = 1200(500 \le \mathrm{h} \le 510)$, it is deemed to have struck the target. To ensure lethal effects, the terminal velocity must exceed 200 m/s to achieve significant penetration depth.

4.3 Uncertain Target Position

When the target position is uncertain, we adopt a fixed muzzle velocity model. For the historical projectile, the muzzle velocity was more difficult to control than the firing angle, as it was primarily determined by the propellant charge. Therefore, the muzzle velocity of the projectile is typically treated as a fixed value, which we set to 450 m/s for this problem. To investigate the relationship between the firing angle and the horizontal distance, we fixed the target altitudes at 500 m and varied the horizontal distance from 1000 to 1500 m in 20-m increments. To examine the effect of the target altitudes, we fixed the horizontal distance at 1200 m and varied the altitudes from 0 to 1000 m in 25-m increments.

4.4 Effects of Wind

Wind is categorized into two types: headwind and crosswind. The actual wind does not reside on a fixed plane.Headwind flows in the same plane as the projectile's trajectory, while crosswind flows perpendicular to this plane. A headwind modifies the relative velocity term in the air resistance formula. This means the drag force is no longer directly opposite to the velocity vector, as its magnitude and direction are now influenced by both the wind's speed and direction. Crosswinds cause the projectile's trajectory to deviate. To maintain accuracy against a stationary target, the gun must be elevated or depressed by a specific angle to compensate for the projectile's deflection during flight.[8]

First, consider the headwind. According to the definition of relative velocity, we obtain: As shown in the figure,where

$$\vec{v}' = \vec{v}_w - \vec{v} \tag{8}$$

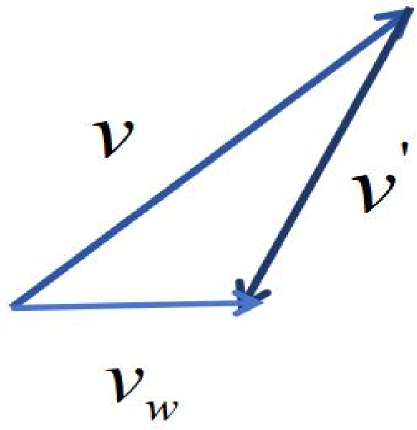

Figure 3. Relative Velocity Vector Synthesis

By analyzing the force diagram of the fireball, we can obtain

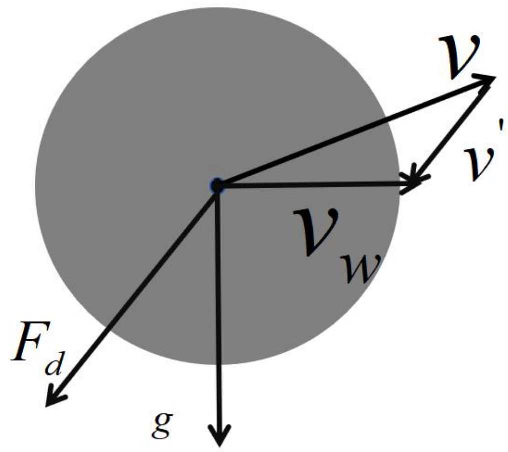

Figure 4. Force Analysis of Projectile Considering Wind Effects

Substituting the actual wind velocity magnitude and direction yields the new equation of motion:

$$\begin{cases} \begin{cases} v''_x = v_0 \cos\theta_0 - \dfrac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v'_x}{2m} t \\ R = \displaystyle\int v'_x dt \end{cases} \\ \begin{cases} v''_y = v_0 \sin\theta_0 - \left(g + \dfrac{C_d \rho_0 e^{-\frac{h}{H_b}} A v v'_y}{2m} \right) t \\ h = \displaystyle\int v'_x dt \end{cases} \end{cases} \tag{9}$$

In practical applications, wind speed is not precisely known. For computational convenience, we assume the wind speed is parallel to the horizontal direction. According to relevant data, wind speeds generally do not exceed 10 m/s. By iterating wind speeds from -10 m/s to 10 m/s, we determine the relationship between the exit angle and wind speed magnitude.[9]

For crosswind. A projectile traveling perpendicular to its trajectory is affected solely by crosswind, experiencing a force of $F_d = \frac{1}{2} c_d A \rho v_{cw}^2$ and a corresponding acceleration of $\frac{c_d A \rho v_{cw}^2}{2m}$.

For uniformly accelerated linear motion with zero muzzle velocity, displacement equals $x = \frac{1}{2} a t^2$. Substituting relevant physical quantities yields a deviation of

$$x_d = \frac{C_d \rho_0 e^{-\frac{h}{H_b}} A v_{cw}^2}{4m} t^2 \quad (10)$$

By sketching a diagram, we can determine the relationship between the deviation angle, firing distance, and deviation distance.

$$\tan \theta_d = \frac{x_d}{R} \quad (11)$$

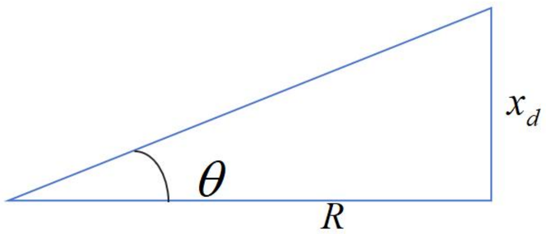

Figure 5. Offset Angle Diagram

4.5 Numerical Methods and Solution Process

The numerical solution of the ordinary differential equations (motion differential equations) is performed using the Dormand-Prince 4(5)th order algorithm (RK45 method) from the explicit Runge-Kutta family. This method achieves adaptive step-size control by simultaneously computing a 4th-order primary estimate and a 5th-order auxiliary estimate. During the integration process, a maximum step size of 0.2 is set to ensure computational stability, and the precision of the numerical solution is controlled by relative tolerance (rtol=1e-6) and absolute tolerance (atol=1e-6).

For solving the firing angle that satisfies the hit condition, the bisection method (Brent's method) is used to search within a preset angle interval. The existence of a root is determined by checking the sign change of the error function at the interval endpoints, and the interval is iteratively narrowed until the angle precision reaches 1e-8 (xtol=1e-8), ultimately obtaining the firing angle that meets the target hit condition.

5 Result and Discussion

5.1. Simple Case

When air resistance is neglected, the expression reveals multiple solutions. Two muzzle firing angles exist: one greater than 45 degrees and one less than 45 degrees.

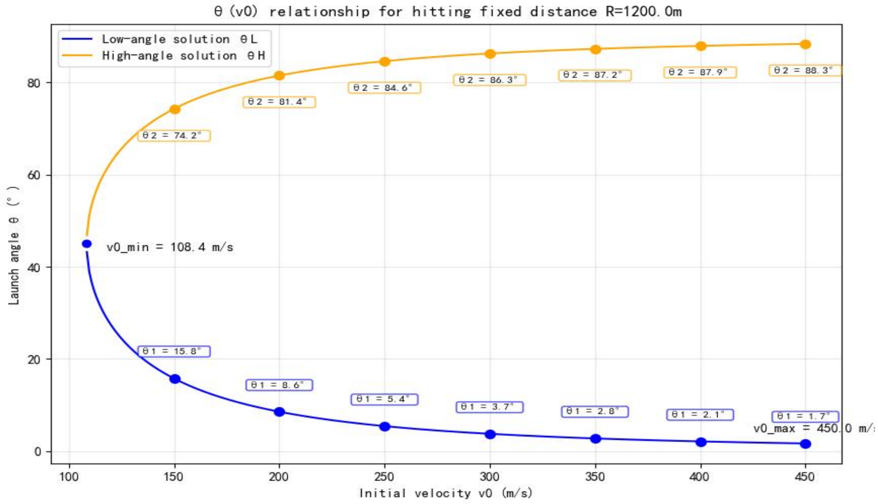

Figure 6. Firing Angle vs. Initial Velocity Relationship Chart

As shown in the figure, the two exit angles exhibit symmetry. When $\theta_L = \theta_H$, the firing velocity is minimal, with a magnitude of 108.4m/s.When the maximum firing velocity $v_0 = 450m/s$ is achieved, θ_1 is at its minimum, 1.7 degrees, and θ_2 is at its maximum, 88.3 degree.

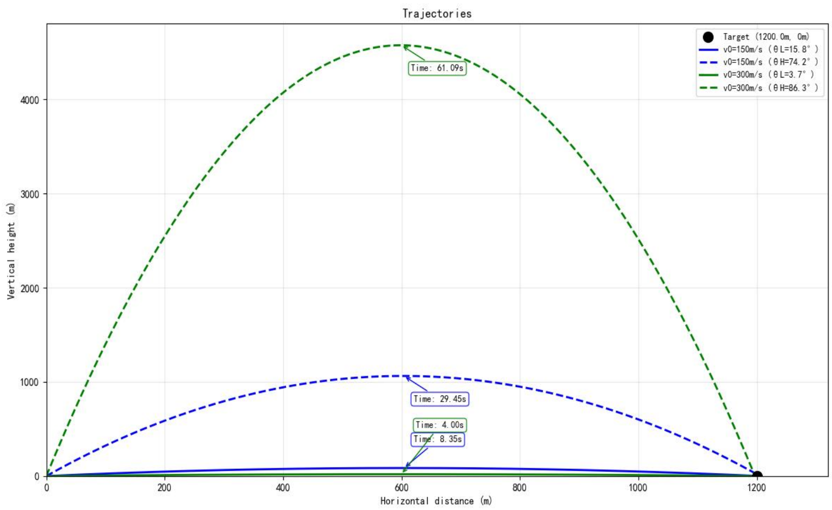

Figure 7. Flight Trajectory

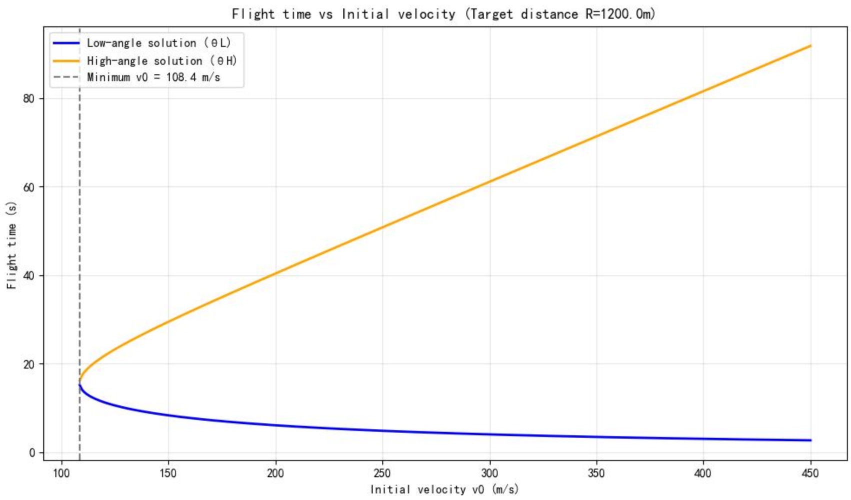

Figure 8. Flight Time vs. Initial Velocity Relationship Chart

As shown in the diagram, when fired at a low angle, a higher muzzle velocity results in a shorter projectile flight time. Conversely, when fired at a high angle, a higher muzzle velocity leads to a longer flight time. This occurs because a steeper firing angle produces a greater maximum altitude for the projectile, thereby extending its airborne duration.

5.2 Considering Air Resistance And Target Size

Taking air resistance into consideration, the projectile's trajectory will no longer be symmetrical but will instead follow a deformed parabola, where the ascent phase covering a greater distance.

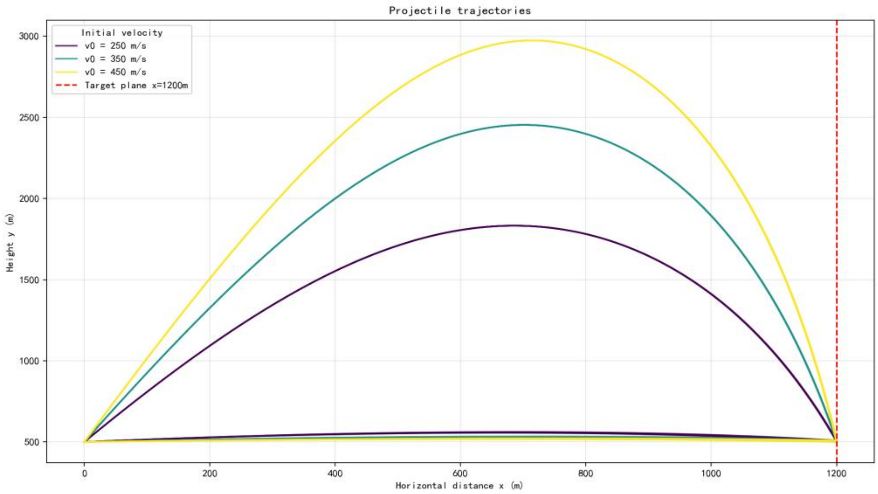

Figure 9. Flight Trajectory

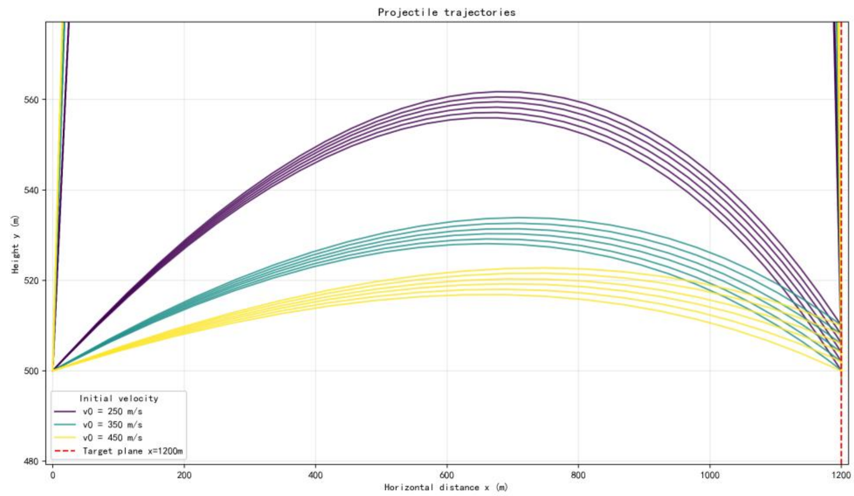

Figure 10. Flight Trajectory (Partial Magnification)

When the target is assumed to have an actual height (set here as 10 m), the firing angle may not be unique at the same firing velocity. In other words, within a certain angular range, the shell can still be considered to have hit the target.

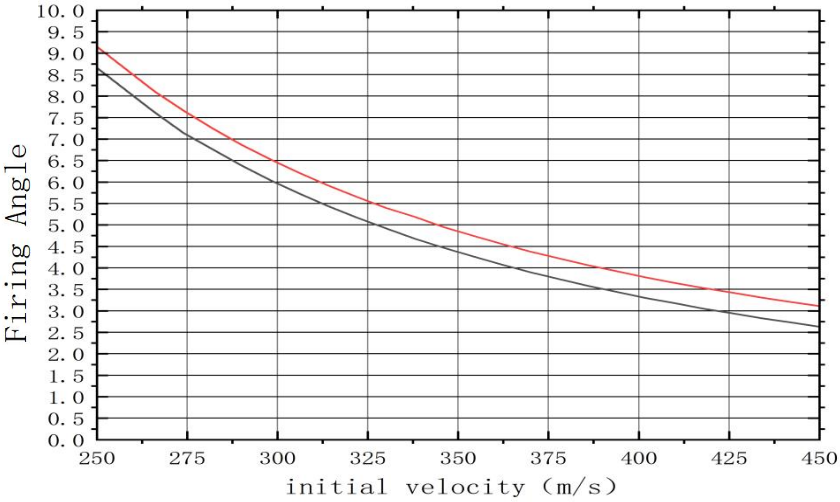

Figure 11. Firing Angle vs. Initial Velocity Chart (Low Angles)

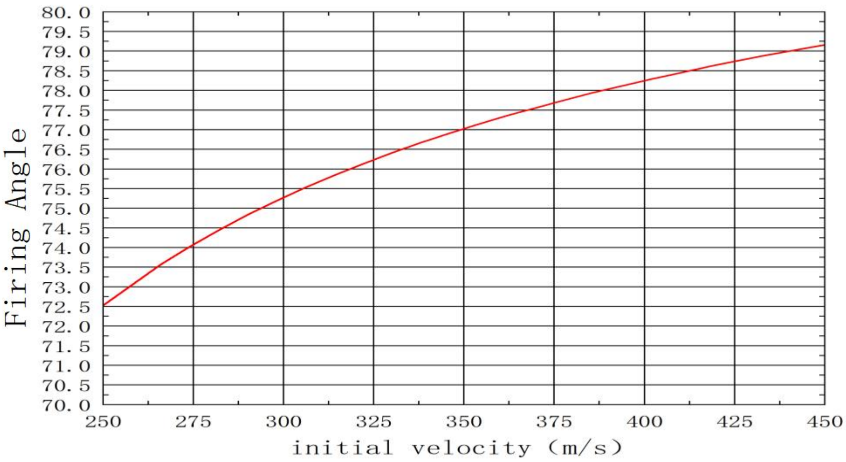

Figure 12. Firing Angle vs. Initial Velocity Diagram (High Angles)

The calculation results show that when firing at a low angle, the projectile can hit the actual target within a range of approximately 0.5 degrees. We employed polynomial fitting to model the relationship between the lower bound of the angular interval and the muzzle velocity, expressed as:

$$\theta_L = 0.0001v_0^2 - 0.1145v_0 + 29.312 \tag{12}$$

The fitting yielded excellent results.At the same time, as the firing velocity increases, the firing angle gradually decreases, similar to the simplified case. However, at the same firing velocity, the angle is relatively larger than the simplified case,it should be to overcome the effects of air resistance.

When firing at high angles, the firing angle becomes nearly unique. This occurs because the projectile's trajectory can only intersect the target's lower point. As the firing velocity increases, the firing angle gradually increases, similar to the simpler case.

5.3 Horizontal distance variation

We fixed the target's altitude at 500m, maintaining the same horizontal line as the cannon, and varied the horizontal distance between the target and the cannon within a range of 1000m to 1500m. The results are as follows:

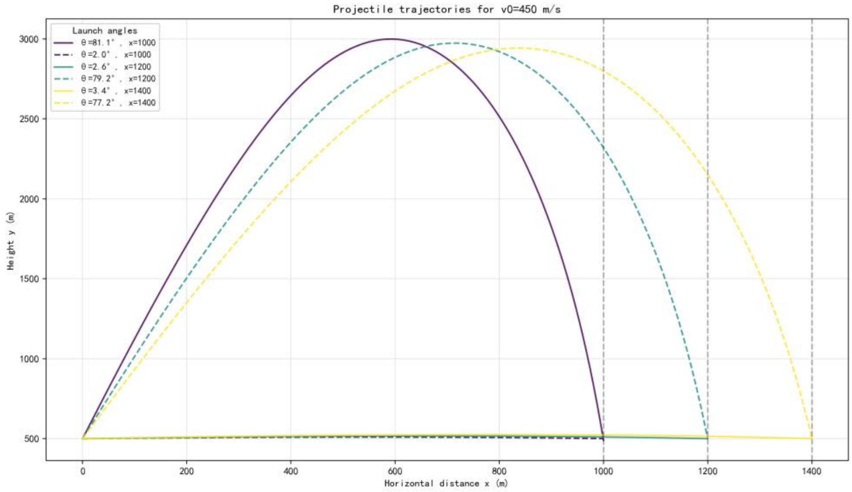

Figure 13. Flight Trajectory

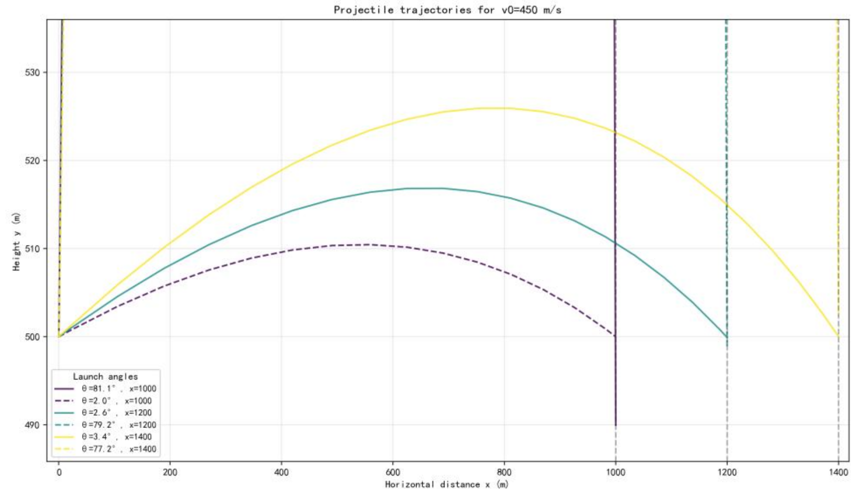

Figure 14. Flight Trajectory (Partial Magnification)

In our calculations, we uniformly adopt an muzzle velocity of 450 m/s, as firing at maximum muzzle velocity is the most common practice when cannon performance is fixed.

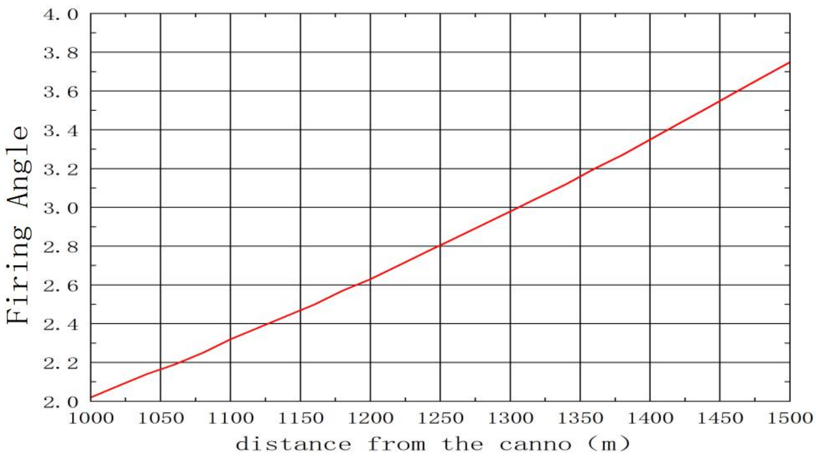

Figure 15. Firing Angle vs. Muzzle Velocity Chart (Low Angles)

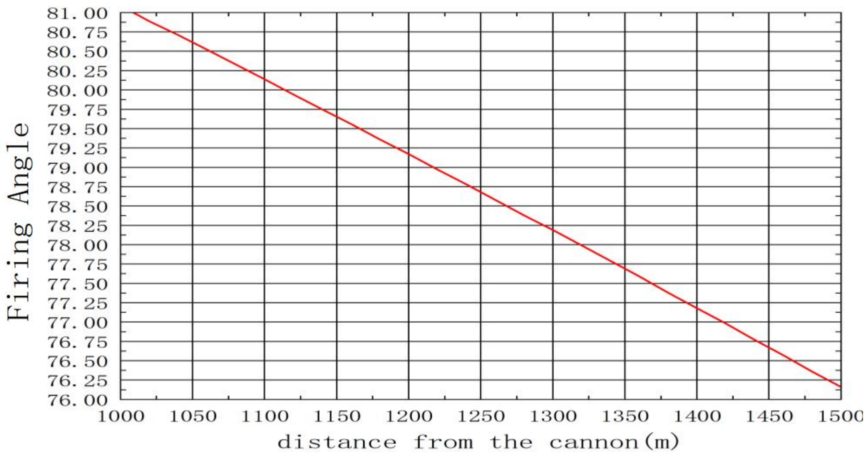

Figure 16. Firing Angle vs. Muzzle Velocity Diagram (High Angles)

As shown in the diagram, when firing at a lower angle, the firing angle must be increased by approximately 0.2 degrees for every 50 meters the target moves away. The firing angle range is between 2 and 3.8 degrees. Achieving this level of precision is feasible with technology from 150 to 200 years ago.

When firing at a high angle, the firing angle must be reduced by approximately 0.2 degrees for every 20 meters the target moves away to ensure a direct hit.

v_0(m/s)	R(m)	t_L(s)	v_L(m/s)	t_H(s)	v_H(m/s)
450	1000	2.9095	268.87	45.6098	136.4559
450	1100	3.2922	255.4526	45.4871	136.3603
450	1200	3.6953	242.7385	45.35	136.2532
450	1300	4.1199	230.6981	45.1977	136.1338
450	1400	4.5675	219.3046	45.0294	136.0012
450	1500	5.0391	208.5333	44.8441	135.8545

Table 2. Terminal velocity and flight time at different horizontal distances

From the table, we can see that when fired at a low angle, the terminal velocity exceeds 200 m/s with a flight time of 3 to 5 seconds. When fired at a high angle, the terminal velocity falls below 140 m/s and the flight time is significantly longer, rendering it incapable of delivering effective lethality. Therefore, in practical scenarios, officers should opt for low-angle firing.

5.4 Vertical height variation

We fixed the target distance for the cannon at 1200m and varied its altitude within the range of 0m to 1000m, yielding the following results:

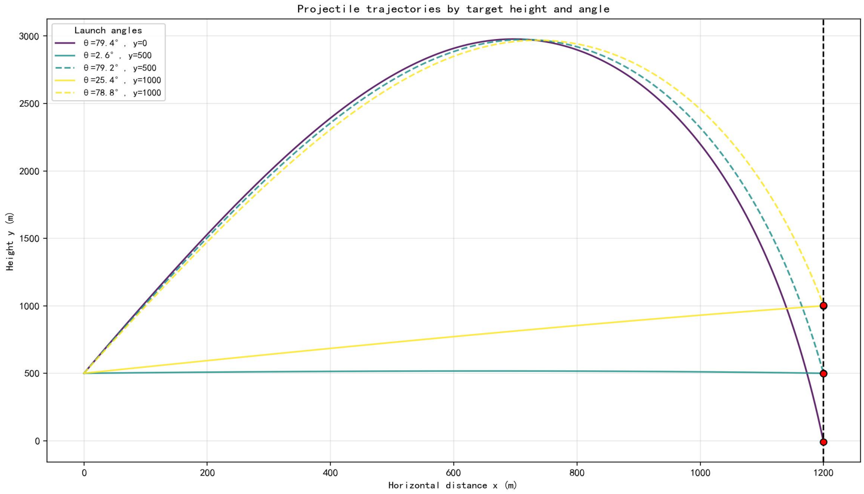

Figure 17. Flight Trajectory

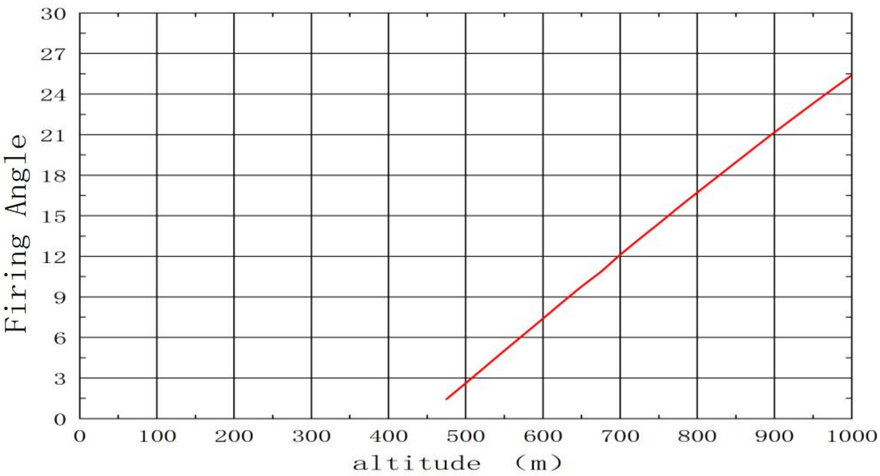

Figure 18. Firing Angle vs. Muzzle Velocity Chart (Low Angles)

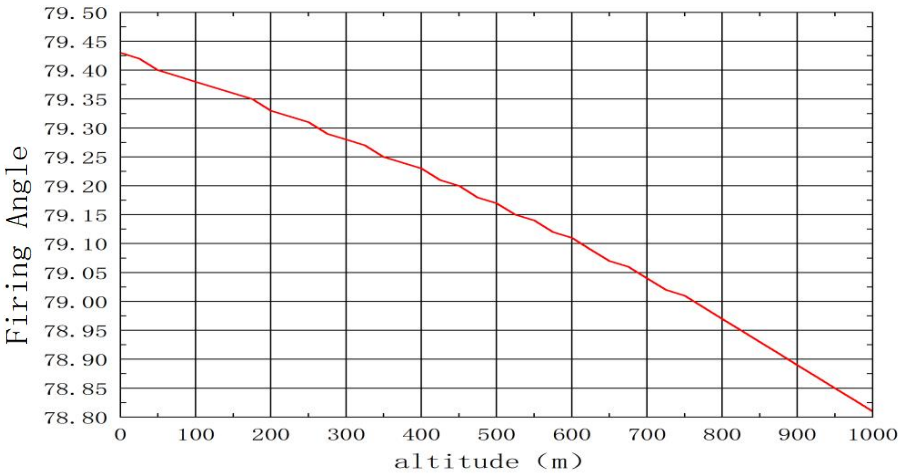

Figure 19. Firing Angle vs. Muzzle Velocity Diagram (High Angles)

As shown in the diagram, when firing at a low angle, the firing angle must increase by approximately 1.1 to 1.2 degrees for every 25-meter rise in target altitude. However, when the target altitude falls below the cannon's altitude, no low-angle solution exists, with the critical altitude ranging between 450 and 475 meters. This occurs because achieving a negative elevation angle—required to hit the target below the firing angle—is impractical.

When firing at a high angle, the elevation angle must increase by approximately 0.02 degrees for every 25-meter rise in target altitude. Achieving this precision was challenging under the technological constraints of the era.

H(m)	t_L (s)	v_L(m/s)	t_H(s)	v_H(m/s)
0	No Solutions		49.0611	136.0033
100			48.3191	136.2044
200			47.5775	136.337
300			46.8358	136.3944
400			46.0935	136.3692
500	3.6953	242.7385	45.35	136.2532
600	3.7234	240.7715	44.6048	136.0376
700	3.7865	237.7598	43.8573	135.7127
800	3.8846	233.7519	43.1066	135.2677
900	4.0175	228.8218	42.3519	134.6909
1000	4.1851	223.061	41.5923	133.9693

Table 3. Terminal velocity and flight time at different heights

Similarly, the data in the table indicates that only the terminal velocity for low-angle firing has practical significance.

5.5 Effects of Wind

When considering wind effects, to simplify calculations, we assume the headwind direction is parallel to the horizontal plane, with speeds ranging from -10 m/s to 10 m/s. A negative sign indicates wind blowing from the target toward the artillery piece.

During the calculation process, we also compute the overall flight time and take its arithmetic mean to provide data support for calculating crosswind drift. When calculating crosswind drift, we assume constant air density and make a simplified estimate: the air density for high-angle launches is 0.94 times that at sea level, while the air density for low-angle launches is 0.82 times that at sea level.

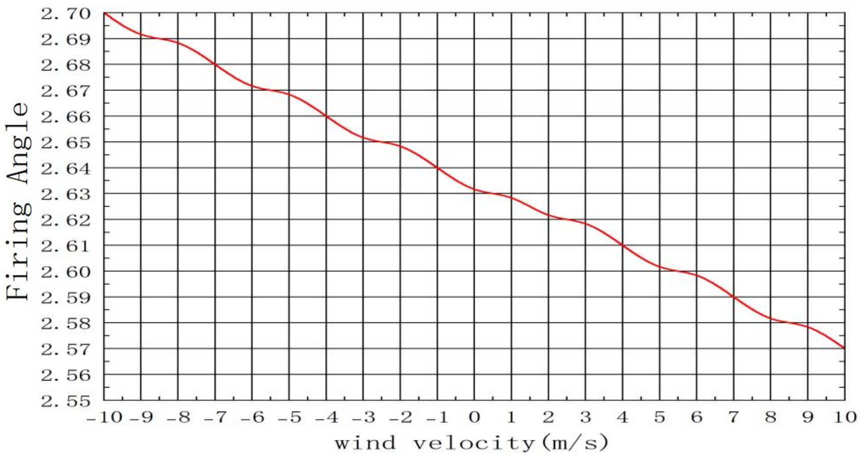

Figure 20. Firing Angle vs. Wind Speed Chart(Low Angles)

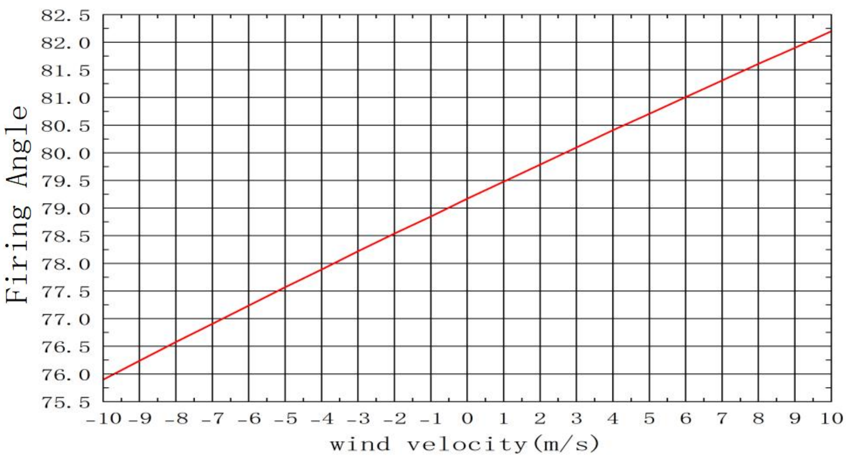

Figure 21. Firing Angle vs. Wind Speed Chart(high Angles)

From the figure, we can see that at low launch angles, parallel wind has a relatively minor impact on the firing trajectory. For every 1 m/s increase in wind speed, the firing angle requires a 0.01-degree offset. This is because at low firing angles, the flight time is too short for wind to significantly affect the trajectory.

However,at high firing angles, however, for every 1 m/s increase in wind speed, the firing angle requires an offset of approximately 0.35 degrees.

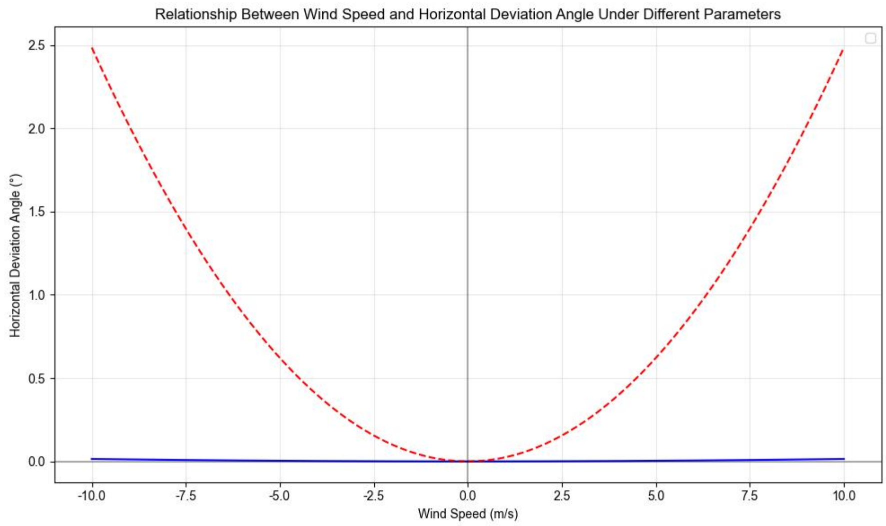

Figure 22. Horizontal Deviation Angle vs Wind speed Chart

When calculating crosswind, we use the above approximation. Simultaneously, calculations show that the flight time for high-angle firing is approximately 45 seconds, while for low-angle firing it is about 3.7 seconds. The crosswind range spans from 0m to 10m. As shown in the figure, during low-angle firing, crosswind drift has negligible impact on the shot—consistent with experience due to the extremely short flight time. Conversely, during high-angle firing, crosswind drift becomes significantly noticeable, requiring real-time adjustments to the cannon's direction based on the crosswind magnitude.

6 Conclusion

Based on the above analysis and calculations, we draw the following conclusion: Under ideal conditions, when a cannon fires at a target point with a given muzzle velocity, there exist two possible firing angles—one large and one small. To ensure sufficient impact energy, projectiles must possess adequate terminal velocity (set at 200 m/s) upon striking the target, subsequent calculations reveal that in multiple scenarios, the terminal velocity achieved at the high angle falls below this threshold. Therefore, we focus solely on the low angle solution. Next, we account for the effects of air resistance by numerically solving the equations of motion. This yields the relationship between launch angle and initial velocity, expressed as the fitting equation: $\theta_L = 0.0001v_0{}^2 - 0.1145v_0 + 29.312$.Considering that objects (such as city walls) typically possess a certain height (set at 10m), thus, the missile can hit the target when the firing angle varies within a range of 0.5° above the calculated value. Next, we calculated cases where the target's horizontal distance and elevation varied within specific ranges. Using a reference point of 1200m horizontal distance and 500m altitudes, with a firing velocity of 450m/s (under sufficient firepower), we found: For every 50m change in horizontal distance, the firing angle must adjust by 0.2°; For every 25m change in altitudes, the firing angle must adjust by 1.1° to 1.2°. It should be noted that when firing at a low angle, targets below the altitudes of the reference point cannot be hit. Finally, considering wind effects, we modify the dynamics equations and solve them. We find: For tailwinds, each 1 m/s increase in wind velocity requires a 0.01° decrease in firing angle; For headwinds, the opposite applies. Since the velocity's flight time is very short during small-angle firing (approximately 3.7 seconds), the deflection angle caused by crosswinds is negligible.

Based on the above conclusions, we have summarized a quick aiming mnemonic for artillery officers. Following this mnemonic, officers need only perform simple addition and subtraction operations along with solving quadratic equations—a piece of cake for officers trained in mathematics. The mnemonics are as follows:

Step 1: Look up the projectile's muzzle velocity by model.

Step 2: Calculate the firing angle to hit the reference point using the fitted formula from the conclusions.

Step 3: For every 25m change in altitudes relative to the reference point, the firing angle changes by 1.1 to 1.2 degrees. For every 50m change in distance, the firing angle changes by 0.2 degrees.

Step 4: Measure the wind velocity. For every 1m/s change in wind velocity, the firing angle changes by 0.01 degrees (decreasing with tailwind, increasing with headwind).

7 Strength and Weakness

7.1 Strength

(1) The model is relatively simple and easy to simulate, comprehensively considers relevant factors, and exhibits minimal error.

(2) Modeling and plotting are performed using Python, enabling intuitive visualization of flight trajectories and computational results under various conditions. The approach progresses from basic to advanced concepts, facilitating reader comprehension.

(3) For different muzzle value problems, fixed variables simplify the problem, streamlining calculations and ensuring clear reasoning.

(4) Practical application considerations, such as ensuring lethality, accounting for target size, and crosswind-induced deflection, provide real-world value.

(5) Provides straightforward, easily understandable conclusions that effectively assist artillery officers in target acquisition.

7.2 weakness

(1) Numerous complex factors within the model were not fully accounted for.For example: Wind velocity may vary in direction and magnitude at different altitudes, significantly complicating the model. To simplify calculations, this factor was excluded.

(2) When calculating air resistance, the drag coefficient was not measured experimentally but estimated based on empirical values, introducing potential errors.

(3) To simplify calculations when target positions are uncertain, one variable is fixed in each direction, limiting the applicability of the final conclusions.

8 Reference

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9 Appendix

9.1 Simple Case

```
import numpy as np
import matplotlib.pyplot as plt
import math

# Plot config
plt.rcParams['font.family'] = ['Arial', 'Helvetica']
plt.rcParams['axes.unicode_minus'] = False

# Calculate θ vs v0 for projectile hitting target R

def calculate_theta_vs_v0(R, g, v0_max):
v0_min = math.sqrt(R * g)
v0_values = np.linspace(v0_min * 1.001, v0_max, 300)
```

```
sin2theta = R * g / (v0_values**2)
theta1 = 0.5 * np.arcsin(sin2theta)  # Low-angle
theta2 = np.pi/2 - theta1            # High-angle

# Flight times
t1 = (2 * v0_values * np.sin(theta1)) / g
t2 = (2 * v0_values * np.sin(theta2)) / g

return {
"v0_min": v0_min, "v0_values": v0_values,
"theta1_deg": np.degrees(theta1), "theta2_deg": np.degrees(theta2),
"t1": t1, "t2": t2, "R": R, "g": g
}

# Plot flight time vs v0
def plot_t_vs_v0_theta(data):
plt.figure(figsize=(10, 6))
plt.plot(data["v0_values"], data["t1"], 'b-', linewidth=2, label='Low-angle (θL)')
plt.plot(data["v0_values"], data["t2"], 'orange', linewidth=2, label='High-angle (θH)')
plt.axvline(x=data["v0_min"], color='gray', linestyle='--',
label=f'Min v0 = {data["v0_min"]:.1f} m/s')
plt.xlabel("Initial velocity v₀ (m/s)")
plt.ylabel("Flight time (s)")
plt.title(f"Flight Time vs v₀ (Target R={data['R']}m)")
plt.grid(alpha=0.3), plt.legend(loc='upper left'), plt.tight_layout()
plt.show()

# Main
if __name__ == "__main__":
R = 1200.0    # Target distance (m)
g = 9.8       # Gravitational acceleration (m/s²)
v0_max = 450.0  # Max initial velocity (m/s)
result_data = calculate_theta_vs_v0(R, g, v0_max)
plot_t_vs_v0_theta(result_data)
```

9.2 Considering Air Resistance

```
import numpy as np
from scipy.integrate import solve_ivp
from scipy.optimize import brentq
import matplotlib.pyplot as plt
import csv
# Core physical parameters
g = 9.81
m = 5.0
d = 0.11
A = np.pi * (d**2) / 4.0
Cd = 0.47
rho0 = 1.225
H = 8500.0
# Initial conditions and target
x0, y0 = 0.0, 500.0
```

```
v0 = 450.0
x_target, y_target = 1200.0, 500
event_x = x_target

def rho(y):
    return rho0 * np.exp(-y / H)

# Differential equations of motion
def deriv(t, state):
    x, y, vx, vy = state
    v = np.hypot(vx, vy)  # Speed magnitude

    if v == 0:
        ax, ay = 0.0, -g
    else:
        # Drag force: Fd = 0.5*Cd*A*rho*v²
        Fd = 0.5 * Cd * A * rho(max(y, 0)) * v**2
        ax = -Fd * vx / (m * v)
        ay = -g - Fd * vy / (m * v)
    return [vx, vy, ax, ay]

# Termination event: stop when reaching target x-coordinate
def event_stop(t, state):
    return state[0] - event_x
event_stop.terminal = True
event_stop.direction = 1

# Error function for angle optimization
def hit_error(angle):
    vx0 = v0 * np.cos(angle)
    vy0 = v0 * np.sin(angle)
    state0 = [x0, y0, vx0, vy0]
    # Solve motion equations
    sol = solve_ivp(
        fun=deriv,
        t_span=[0, 500],
        y0=state0,
        events=event_stop,
        max_step=0.2,
        rtol=1e-6, atol=1e-6
    )
    x_end = sol.y[0, -1]
    y_end = sol.y[1, -1]
    x_error = (x_end - x_target) / x_target if x_target != 0 else 0
    y_error = (y_end - y_target) / (abs(y_target) + 1)
    return x_error + y_error

# Find valid launch angles to hit target
def solve_hit_angle(angle_min_deg=1, angle_max_deg=85, n_scan=500):
    angles_rad = np.radians(np.linspace(angle_min_deg, angle_max_deg, n_scan))
    errors = [hit_error(angle) for angle in angles_rad]
    valid_angles = []
```

```
    # Find sign changes in error (indicates root)
    for i in range(len(errors) - 1):
        if errors[i] * errors[i + 1] < 0:
            try:
                # Refine angle with Brent's method
                angle_rad = brentq(hit_error, angles_rad[i], angles_rad[i + 1], xtol=1e-8)
                vx0 = v0 * np.cos(angle_rad)
                vy0 = v0 * np.sin(angle_rad)
                    sol = solve_ivp(deriv, [0, 500], [x0, y0, vx0, vy0], events=event_stop,
max_step=0.2)
                x_end, y_end = sol.y[0, -1], sol.y[1, -1]
                if abs(x_end-x_target) < 5 and abs(y_end-y_target)< 1:
                    valid_angles.append(round(np.degrees(angle_rad),2))
            except ValueError:
                continue

     return list(set(valid_angles))  # Remove duplicates

if __name__ == "__main__":
    hit_angles = solve_hit_angle()
```

9.3 Effects of Wind

```
import numpy as np
from scipy.optimize import brentq
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
import csv

# Core physical parameters
g = 9.81
m = 5.0
d = 0.11
A = np.pi * (d**2) / 4.0
Cd = 0.47
rho0 = 1.225
H = 8500.0
# Launch & target conditions
x0, y0 = 0.0, 500.0
v0 = 450.0
x_target, y_target = 1200.0, 500.0
event_x = x_target
def rho(y):
    return rho0 * np.exp(-y / H)
def event_stop(t, state):
    return state[0] - event_x
event_stop.terminal = True
event_stop.direction = 1

# Solve valid launch angles for given derivative function
def solve_hit_angle_custom_deriv(deriv_func, angle_min_deg=1,
angle_max_deg=85, n_scan=200):
```

```
    angles_rad = np.radians(np.linspace(angle_min_deg, angle_max_deg, n_scan))
    errors = [hit_error_custom_deriv(a, deriv_func) for a in angles_rad]

    valid_angles = []
    for i in range(len(errors) - 1):
        if errors[i] * errors[i+1] < 0:
            try:
                angle_rad = brentq(lambda a: hit_error_custom_deriv(a, deriv_func),
                                   angles_rad[i], angles_rad[i+1], xtol=1e-8)
                # Validate solution
                vx0, vy0 = v0*np.cos(angle_rad), v0*np.sin(angle_rad)
                 sol = solve_ivp(deriv_func, [0, 500], [x0, y0, vx0, vy0], events=event_stop,
max_step=0.2)
                x_end, y_end = sol.y[0, -1], sol.y[1, -1]
                if abs(x_end - x_target) < 5 and abs(y_end-y_target)<1:
                    valid_angles.append(round(np.degrees(angle_rad),2))
            except ValueError:
                continue
    return list(set(valid_angles))

# Error function for angle optimization
def hit_error_custom_deriv(angle, deriv_func):
    vx0, vy0 = v0*np.cos(angle), v0*np.sin(angle)
      sol  =  solve_ivp(deriv_func,  [0,  500],  [x0,  y0,  vx0,  vy0],  events=event_stop,
max_step=0.2)
    return sol.y[1, -1] - y_target

# Compute angles across wind speed range
def compute_angles_vs_wind(Vw_list):
    angle_wind = []
    for Vw in Vw_list:
        # Motion equations with horizontal wind
        def deriv_wind(t, state):
            x, y, vx, vy = state
            vx_rel = vx - Vw  # Relative velocity (projectile - wind)
            vy_rel = vy
            v_rel = np.hypot(vx_rel, vy_rel)
            if v_rel == 0:
                ax, ay = 0.0, -g
            else:
                Fd = 0.5 * Cd * A * rho(max(y, 0)) * v_rel**2
                ax = -Fd * vx_rel / (m * v_rel)
                ay = -g - Fd * vy_rel / (m * v_rel)
            return [vx, vy, ax, ay]
        angles = solve_hit_angle_custom_deriv(deriv_wind)
        angle_wind.append(angles)
        print(f"Wind speed Vw={Vw:.1f} m/s --> Launch angles: {angles}")
    return angle_wind

# Main execution
if __name__ == "__main__":
    # Wind speed range: -10 to 10 m/s (21 points)
```

```
Vw_list = np.linspace(-10, 10, 21)
angle_wind = compute_angles_vs_wind(Vw_list)
```