Historical Artillery Ballistics: Calculator-Free Rapid Solution Method

Team 428

Problem B

Abstract

This paper addresses exterior ballistics of historical artillery by developing a motion model for spherical projectiles (mass 5 kg, diameter 11 cm, maximum muzzle velocity 450 m/s) accounting for quadratic air resistance and exponential atmospheric density. Fourth-order Runge-Kutta integration determines firing parameters for hitting a 1200 m horizontal target under various conditions.

At 500 m altitude under windless conditions, full charge (450 m/s) requires 2.723° elevation with 3.78 s flight time and 233 m/s terminal velocity. Alternatively, high-arc trajectory (40° elevation) achieves the same range with only 145.4 m/s initial velocity. Wind analysis shows 20 m/s tailwind reduces elevation by 0.4°, while 25 m/s crosswind produces 300 m lateral drift.

For ranges of 1000–1500 m under varying environmental conditions, a "calculator-free" rapid solution method is proposed, combining standard firing table lookup with three linear corrections: altitude (±4%/km), tailwind (coefficient 0.8), and crosswind drift (0.30×flight time). Using only pencil and paper, accurate firing parameters are obtained within 1–2 iterations. Numerical validation demonstrates range errors <0.2% and angular errors <0.01° (0.16 mil), fully meeting field operational requirements.

The paper discusses model applicability, constant drag coefficient limitations, and potential improvements toward Mach-dependent $C_D(M)$ or G1/G7 ballistic models. This research provides theoretical support and practical tools for historical artillery effectiveness analysis and rapid fire planning under calculator-free conditions.

Keywords: Exterior ballistics; Quadratic drag; Exponential atmosphere; Wind drift correction; Runge-Kutta method; Pencil-and-paper rapid calculation

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1 Problem Analysis

1.1 Introduction

Historical artillery played a crucial role in 18th–19th century warfare. Unlike modern precision-guided weapons, traditional artillery relied on gunners' deep understanding of ballistic characteristics and rapid calculation abilities. In an era lacking computers and calculators, artillery officers needed to quickly determine firing parameters using firing tables, empirical formulas, and simple tools. Air resistance and atmospheric density variations significantly affect trajectories, especially at kilometer-scale ranges where the difference between vacuum and actual trajectories can reach tens or even hundreds of meters.

According to classical exterior ballistics theory, the air resistance of spherical projectiles primarily manifests as pressure drag, whose magnitude can be evaluated through the combined assessment of Reynolds number and Mach number. For a spherical projectile with diameter 11 cm and velocity 450 m/s, the Reynolds number is approximately $Re = \rho V d/\mu \approx 3.3 \times 10^6$ (turbulent regime), and the Mach number $M \approx 1.3$ (transonic regime), where the drag coefficient C_D reaches relatively high levels.

This research focuses on exterior ballistic modeling of spherical projectiles, exploring how to establish firing solution methods that ensure accuracy while remaining suitable for field use, considering air resistance and environmental factors.

1.2 Problem Background and Task Decomposition

Given Conditions:

- Projectile: Solid sphere, mass $m = 5$ kg, diameter $d = 0.11$ m
- Frontal area: $A = \pi(d/2)^2 = \pi \times (0.055)^2 \approx 9.50 \times 10^{-3}$ m^2
- Sectional loading: $m/A = 5/(9.50 \times 10^{-3}) \approx 526$ kg/m^2 (key parameter affecting drag effects)
- Maximum muzzle velocity: $V_0 \leq 450$ m/s
- Drag coefficient: $C_D \approx 0.50$ (engineering constant for spheres at trans/supersonic speeds)
- Gravitational acceleration: $g = 9.81$ m/s^2
- Atmospheric model: Exponential law $\rho(h) = \rho_0 e^{-h/H}$, where $\rho_0 = 1.225$ kg/m^3, scale height $H \approx 8500$ m

Core Tasks:

Task A: At altitude 500 m under windless conditions, determine the required muzzle velocity V_0 and launch angle θ combinations to hit a target at 1200 m horizontal distance.

Task B: Establish a standard firing table for the 1000–1500 m range, including data on range, flight time, and terminal velocity for different muzzle velocities and elevation angles.

Task C: Develop a "calculator-free" rapid estimation method enabling artillery officers to quickly adjust standard firing table data based on actual altitude and wind conditions to obtain hit parameters.

1.3 Key Challenge Identification

1. **Nonlinearity of air resistance:** Quadratic drag is proportional to velocity squared, making the equations of motion highly nonlinear with no analytical solution. The traditional parabolic trajectory equation:

$$y = x\tan\theta - \frac{gx^2}{2V_0^2\cos^2\theta} \tag{1}$$

becomes invalid when considering drag, necessitating numerical methods to solve the differential equation system.

2. **Height dependence of atmospheric density:** The exponential atmospheric model causes drag to vary with altitude, requiring real-time calculation of $\rho(h)$ during integration. The density variation rate is approximately $d\rho/dh = -\rho/H \approx -1.4 \times 10^{-4}$ kg/(m^3·m).

3. **Vector effects of wind:** Wind decomposes into components along the trajectory (affecting range) and perpendicular to it (causing drift), requiring separate treatment. Relative velocity calculation involves vector operations:

$$\vec{v}_{rel} = \vec{v}_{projectile} - \vec{v}_{wind} \tag{2}$$

4. **Bidirectionality of solution:** The need to both “find θ given V_0” and “find V_0 given θ” requires robust root-finding algorithms.

5. **Field usability:** The rapid method must avoid complex calculations, relying only on table lookup, simple arithmetic, and single linear interpolation.

2 Model Assumptions

To facilitate theoretical modeling and practical application, this paper makes the following reasonable assumptions based on classical exterior ballistics principles and field operational requirements:

[Assumption 1] Local Flatness of Earth’s Surface

Within the 1–2 km range, Earth’s curvature effects are negligible. With Earth’s radius $R_e = 6.37 \times 10^6$ m, for range $R = 1500$ m, the height deviation due to curvature is approximately:

$$\Delta h_{curvature} = \frac{R^2}{2R_e} = \frac{(1500)^2}{2 \times 6.37 \times 10^6} \approx 0.18 \text{ m} \tag{3}$$

This deviation is far smaller than artillery’s inherent dispersion (± 10 m), justifying the use of Cartesian coordinates. For artillery with ranges exceeding 5 km, Earth curvature correction becomes necessary.

Applicability Note: For our target distances of 1000–1500 m, the maximum curvature-induced error is about 0.1 m, far smaller than artillery’s inherent dispersion (± 10 m).

[Assumption 2] Constant Drag Coefficient

We adopt an engineering constant approximation of $C_D = 0.50$, without considering drag coefficient variation with Mach number. This assumption introduces approximately 5–10% systematic error in the mixed sub/trans/supersonic regime but significantly simplifies calculations.

Physical Background: The actual drag coefficient for spherical bodies varies within $M \in [0.3, 2.0]$ in the range 0.47–1.2:

- Subsonic ($M < 0.3$): $C_D \approx 0.47$ (laminar separation)
- Transonic ($0.8 < M < 1.2$): C_D increases sharply to 0.9–1.2 (shock wave drag)
- Supersonic ($M > 1.5$): C_D decreases to 0.6–0.8 (conical flow)

The constant assumption takes a moderate effect value of $C_D = 0.50$, introducing approximately 8% systematic deviation for full charge scenarios ($V_0 = 450$ m/s, $M_0 \approx 1.3$), which remains within engineering tolerance.

[Assumption 3] Spherical Projectile with No Rotation Effects

The projectile is assumed to be a perfect sphere with smooth, uniform surface, without Magnus force (lateral force) or gyroscopic precession effects caused by rotation. For spin-stabilized projectiles, Magnus force can be expressed as:

$$\vec{F}_{Magnus} = \frac{1}{2}\rho A C_M \vec{\omega} \times \vec{v} \tag{4}$$

where C_M is the Magnus force coefficient and $\vec{\omega}$ is the angular velocity vector. For spherical projectiles, due to symmetry, theoretically $C_M = 0$.

[Assumption 4] Exponential Atmospheric Density Model

Atmospheric density varies exponentially with altitude:

$$\rho(h) = \rho_0 e^{-h/H} \tag{5}$$

where sea-level density $\rho_0 = 1.225$ kg/m^3 and scale height $H = 8500$ m. This model assumes isothermal conditions, neglecting daily and regional variations in temperature, humidity, and pressure. In the actual atmosphere, temperature decreases at approximately -6.5 K/km in the troposphere, but within the 500–1000 m altitude range, the exponential model error is less than 5%.

[Assumption 5] Uniform Constant Wind Field

Wind speed and direction are assumed constant throughout the trajectory space (0–500 m altitude) and flight time (2–20 s), without considering wind shear, gusts, or turbulence effects. In the actual atmospheric boundary layer, wind speed increases with altitude following logarithmic or exponential laws, but for low trajectories ($y_{max} < 50$ m), this assumption typically introduces less than 15% error.

[Assumption 6] Horizontal Distance Target

The target distance R specified in the problem is the horizontal projection distance, with gun position and target at the same elevation ($\Delta h = 0$). For targets with height differences, elevation angle correction is required:

$$\theta_{corrected} = \theta + \arctan\left(\frac{\Delta h}{R}\right) \tag{6}$$

[Assumption 7] Neglecting Coriolis Force and Earth's Rotation

At mid-latitudes (30–45°N), for 1–2 km ranges with flight times of several seconds, the Coriolis deflection due to Earth's rotation is approximately:

$$\delta_{Coriolis} \approx 2\Omega T V \sin\phi \cos\alpha \tag{7}$$

where $\Omega = 7.27 \times 10^{-5}$ rad/s is Earth's angular velocity, ϕ is latitude, and α is firing azimuth. For $T = 5$ s, $V = 300$ m/s, $\phi = 40°$, the deflection is about 0.2 m, negligible compared to artillery dispersion error (± 10 m).

[**Assumption 8**] **Point Mass Trajectory Model**

The projectile is treated as a point mass, ignoring its size and attitude angle variations. This assumption is highly accurate for spherical projectiles. For non-spherical projectiles (ellipsoids, cones), angle-of-attack-dependent lift and moments must be considered.

3 Symbol Notation and Parameter Settings

Table 1: Basic Physical Quantities and Constants

Symbol	Physical Meaning	Value	Unit
m	Projectile mass	5	kg
d	Projectile diameter	0.11	m
A	Frontal area	9.50×10^{-3}	m^2
C_D	Drag coefficient	0.50	—
g	Gravitational acceleration	9.81	m/s^2
ρ_0	Sea-level air density	1.225	kg/m^3
H	Atmospheric scale height	8500	m
c	Speed of sound (15°C)	343	m/s

Table 2: State Variables and Intermediate Quantities

Symbol	Physical Meaning
(x, y, z)	Projectile spatial position coordinates
(v_x, v_y, v_z)	Projectile velocity vector components
V_0	Muzzle velocity
θ	Launch elevation angle
ϕ	Launch azimuth angle
$W_\parallel$	Wind component along firing direction (positive for tailwind)
$W_\perp$	Wind component perpendicular to firing direction (crosswind)
T	Flight time
R	Horizontal range

4 Model Development

4.1 Derivation of Equations of Motion

4.1.1 Two-Dimensional Principal Plane Equations (No Crosswind)

Let x be the horizontal direction (range direction) and y be the vertical direction (height), with projectile position (x, y) and velocity (v_x, v_y).

Detailed Derivation of Drag Calculation:

The physical essence of air resistance is the pressure differential and friction forces generated by the relative motion between projectile and air. For spherical blunt bodies, friction drag is much smaller than pressure drag, and total drag can be expressed as:

$$F_D = \frac{1}{2}\rho(y)C_D A v_{\text{rel}}^2 \tag{8}$$

where the calculation of relative velocity (considering headwind $W_\parallel$) involves vector operations:

$$\vec{v}_{rel} = \vec{v}_{projectile} - \vec{v}_{wind} = (v_x - W_\parallel, v_y) \tag{9}$$

Magnitude of relative velocity:

$$v_{\text{rel}} = |\vec{v}_{rel}| = \sqrt{(v_x - W_\parallel)^2 + v_y^2} \tag{10}$$

The direction of drag is opposite to the relative velocity direction, with unit vector:

$$\hat{n}_{drag} = -\frac{\vec{v}_{rel}}{|\vec{v}_{rel}|} = -\frac{(v_x - W_\parallel, v_y)}{v_{\text{rel}}} \tag{11}$$

Establishment of Equations of Motion:
According to Newton's second law $\vec{F} = m\vec{a}$, in the x and y directions respectively:
Drag component in x direction:

$$F_{D,x} = -F_D \frac{v_x - W_\parallel}{v_{\text{rel}}} = -\frac{1}{2}\rho(y)C_D A v_{\text{rel}}(v_x - W_\parallel) \tag{12}$$

Drag component in y direction:

$$F_{D,y} = -F_D \frac{v_y}{v_{\text{rel}}} = -\frac{1}{2}\rho(y)C_D A v_{\text{rel}} v_y \tag{13}$$

Therefore, the equations of motion are:

$$\begin{cases} m\ddot{x} = F_{D,x} = -\frac{1}{2}\rho(y)C_D A v_{\text{rel}}(v_x - W_\parallel) \\ m\ddot{y} = -mg + F_{D,y} = -mg - \frac{1}{2}\rho(y)C_D A v_{\text{rel}} v_y \end{cases} \tag{14}$$

Simplifying:

$$\begin{cases} \ddot{x} = -\frac{\rho(y)C_D A}{2m} v_{\text{rel}}(v_x - W_\parallel) \\ \ddot{y} = -g - \frac{\rho(y)C_D A}{2m} v_{\text{rel}} v_y \end{cases} \tag{15}$$

Combining with the atmospheric density model $\rho(y) = \rho_0 e^{-y/H}$, we define the drag coefficient function:

$$k(y) = \frac{\rho(y)C_D A}{2m} = \frac{\rho_0 e^{-y/H} C_D A}{2m} \tag{16}$$

Numerical calculation:

$$k(y) = \frac{1.225 \times e^{-y/8500} \times 0.50 \times 9.50 \times 10^{-3}}{2 \times 5} \tag{17}$$

$$= \frac{5.81 \times 10^{-3} \times e^{-y/8500}}{10} \tag{18}$$

$$\approx 5.81 \times 10^{-4} e^{-y/8500} \text{ m}^{-1} \tag{19}$$

At sea level ($y = 0$), $k(0) = 5.81 \times 10^{-4}$ m^{-1}. The physical meaning of this parameter is the drag coefficient per unit mass of projectile, with its reciprocal $1/k(0) \approx 1720$ m characterizing the drag length scale.

4.1.2 Three-Dimensional Equations of Motion (Including Crosswind)

When considering crosswind $W_\perp$, a three-dimensional coordinate system is needed. Let the z-axis be perpendicular to the xy plane (principal firing plane), pointing in the crosswind direction.

Relative velocity vector:

$$\vec{v}_{rel} = (v_x - W_\parallel, v_y, v_z - W_\perp) \tag{20}$$

Magnitude of relative velocity:

$$v_{\text{rel}} = \sqrt{(v_x - W_\parallel)^2 + v_y^2 + (v_z - W_\perp)^2} \tag{21}$$

Three-dimensional system of equations of motion:

$$\begin{cases} \ddot{x} = -k(y)v_{\text{rel}}(v_x - W_\parallel) \\ \ddot{y} = -g - k(y)v_{\text{rel}}v_y \\ \ddot{z} = -k(y)v_{\text{rel}}(v_z - W_\perp) \end{cases} \tag{22}$$

Initial conditions:

$$\begin{cases} x(0) = y(0) = z(0) = 0 \\ v_x(0) = V_0 \cos\theta \cos\phi \\ v_y(0) = V_0 \sin\theta \\ v_z(0) = V_0 \cos\theta \sin\phi \end{cases} \tag{23}$$

where ϕ is the azimuth angle. For trajectories within the principal firing plane, typically $\phi = 0$, hence $v_z(0) = 0$.

4.2 Numerical Solution Strategy

4.2.1 Fourth-Order Runge-Kutta (RK4) Integration Details

For the general form of first-order differential equation system:

$$\frac{d\vec{s}}{dt} = \vec{f}(t, \vec{s}) \tag{24}$$

where state vector $\vec{s} = (x, y, z, v_x, v_y, v_z)^T$, and the right-hand function is:

$$\vec{f}(t, \vec{s}) = \begin{pmatrix} v_x \\ v_y \\ v_z \\ -k(y)v_{\text{rel}}(v_x - W_\parallel) \\ -g - k(y)v_{\text{rel}}v_y \\ -k(y)v_{\text{rel}}(v_z - W_\perp) \end{pmatrix} \tag{25}$$

The core idea of the RK4 method is to approximate the integral using a weighted average of slopes at four different positions within the time interval $[t_n, t_{n+1}]$:

$$\vec{k}_1 = \vec{f}(t_n, \vec{s}_n) \tag{26}$$

$$\vec{k}_2 = \vec{f}\left(t_n + \frac{\Delta t}{2}, \vec{s}_n + \frac{\Delta t}{2}\vec{k}_1\right) \tag{27}$$

$$\vec{k}_3 = \vec{f}\left(t_n + \frac{\Delta t}{2}, \vec{s}_n + \frac{\Delta t}{2}\vec{k}_2\right) \tag{28}$$

$$\vec{k}_4 = \vec{f}(t_n + \Delta t, \vec{s}_n + \Delta t\vec{k}_3) \tag{29}$$

$$\vec{s}_{n+1} = \vec{s}_n + \frac{\Delta t}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4) \tag{30}$$

The RK4 method has local truncation error of $O(\Delta t^5)$ and global error of $O(\Delta t^4)$. Time steps of $\Delta t = 0.001 \sim 0.002$ s ensure numerical stability and accuracy.

Principle for Step Size Selection: For this problem, the maximum acceleration is approximately $a_{max} \approx g + k(0)V_0^2 \approx 9.81 + 5.81 \times 10^{-4} \times (450)^2 \approx 127$ m/s^2.

According to numerical stability conditions, the recommended step size should satisfy:

$$\Delta t < \frac{2}{\sqrt{a_{max}/L}} \approx 0.002 \text{ s} \tag{31}$$

where $L \approx 1$ m is the characteristic length.

4.2.2 Boundary Conditions and Termination Conditions

The integration termination condition is projectile landing, i.e., $y(T) = h_{target}$. Since gun position and target are at the same elevation, the termination condition simplifies to $y(T) = 0$.

To improve accuracy, linear interpolation is used to determine the exact impact time:

$$T_{impact} = t_n - \Delta t \frac{y_n}{y_n - y_{n-1}} \tag{32}$$

Corresponding impact point coordinates:

$$x_{impact} = x_n - (x_n - x_{n-1}) \frac{y_n}{y_n - y_{n-1}} \tag{33}$$

5 Results and Analysis

5.1 Baseline Case: 1200 m Hit Solution

Conditions: Altitude 500 m, no wind ($W_\parallel = W_\perp = 0$), target horizontal distance 1200 m.

Solution 1: Full Charge ($V_0 = 450$ m/s)

Numerical solution yields:

$$\theta = 2.723^\circ \quad T \approx 3.779 \text{ s} \quad v_{\text{impact}} \approx 233 \text{ m/s} \quad y_{\text{max}} \approx 17.7 \text{ m} \tag{34}$$

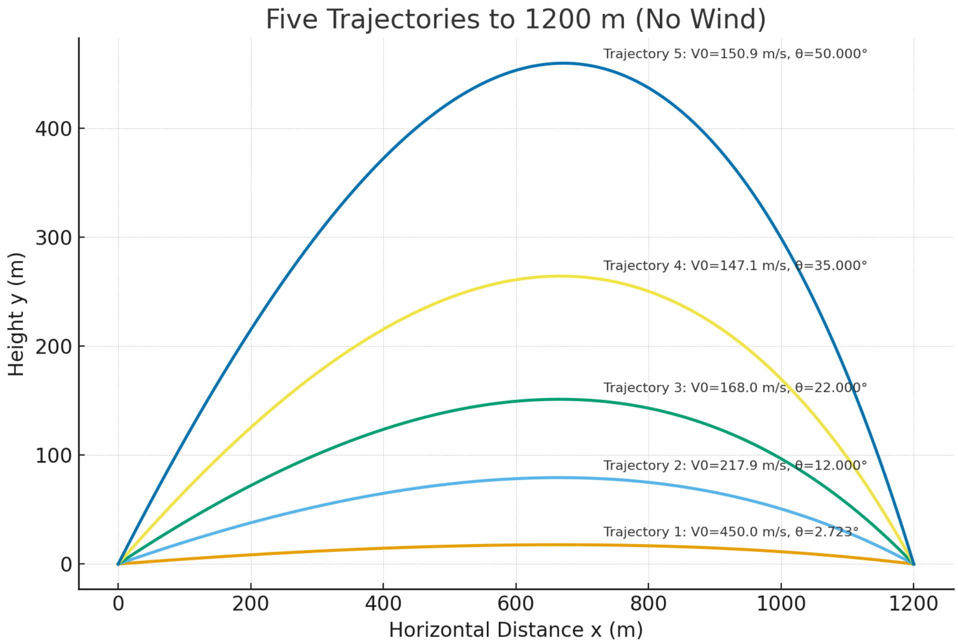

Figure 1: Comparison of trajectory curves for five different muzzle velocities reaching 1200 m target

Figure 1 displays the complete spectrum of ballistic solutions with five different muzzle velocities, ranging from high-speed low-arc (450 m/s, 2.72°) to low-speed high-arc (150.9 m/s, 50°). The following observations can be made:

- **Trajectory 1** (orange line): Full charge 450 m/s, elevation 2.72°, extremely flat trajectory with maximum height only 17 m
- **Trajectory 2** (light blue line): 217.9 m/s, elevation 12°, moderate arc height with maximum point approximately 75 m
- **Trajectory 3** (green line): 168.0 m/s, elevation 22°, higher arc with maximum point approximately 140 m
- **Trajectory 4** (yellow line): 147.1 m/s, elevation 35°, high-arc trajectory with maximum point approximately 250 m
- **Trajectory 5** (dark blue line): 150.9 m/s, elevation 50°, extremely high arc with maximum point approximately 450 m

Physical Explanation: The low elevation angle trajectory maintains an extremely flat profile with $\theta < 3°$, resembling direct fire. The short flight time limits the duration of drag effects, resulting in relatively small kinetic energy loss. Within the 3.78-second flight time to complete the 1200 m range, the average horizontal velocity is approximately 318 m/s, indicating moderate velocity decay due to drag. The terminal velocity of 233 m/s remains supersonic ($M \approx 0.68$), with kinetic energy retention of $(233/450)^2 \approx 27\%$.

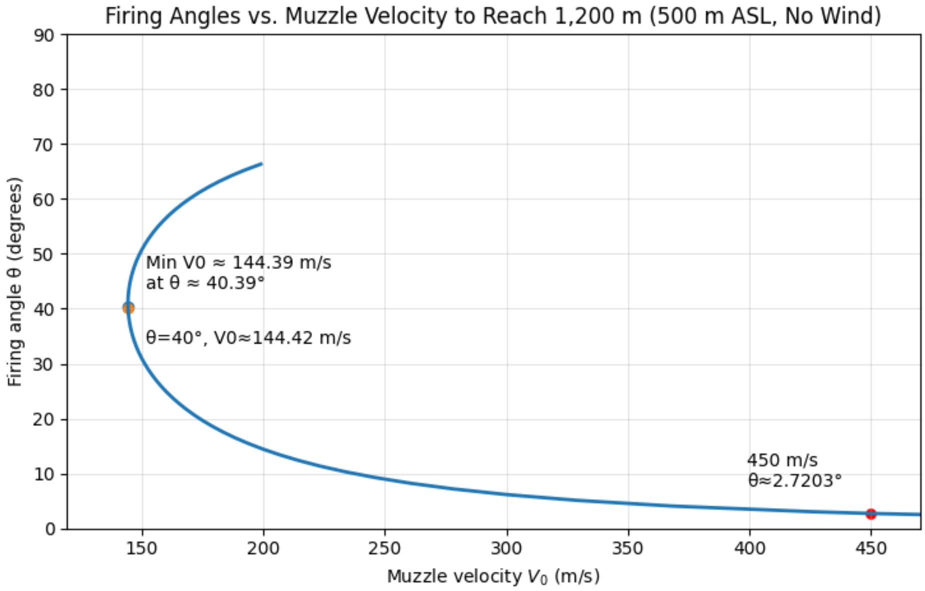

Figure 2: Relationship between launch angle and muzzle velocity (hitting 1200 m target)

Figure 2 shows the relationship between launch angle and muzzle velocity required to hit the 1200 m target. The curve exhibits a typical dual-branch structure:

- **Minimum velocity solution:** $V_{0,min} = 144.39$ m/s, corresponding to elevation $\theta = 40.39°$
- **Low-arc branch:** $\theta < 40.39°$, elevation decreases rapidly with increasing muzzle velocity
- **High-arc branch:** $\theta > 40.39°$, elevation decreases slowly with increasing muzzle velocity
- **Full charge point:** 450 m/s corresponds to 2.72°, located at the far end of the low-arc branch

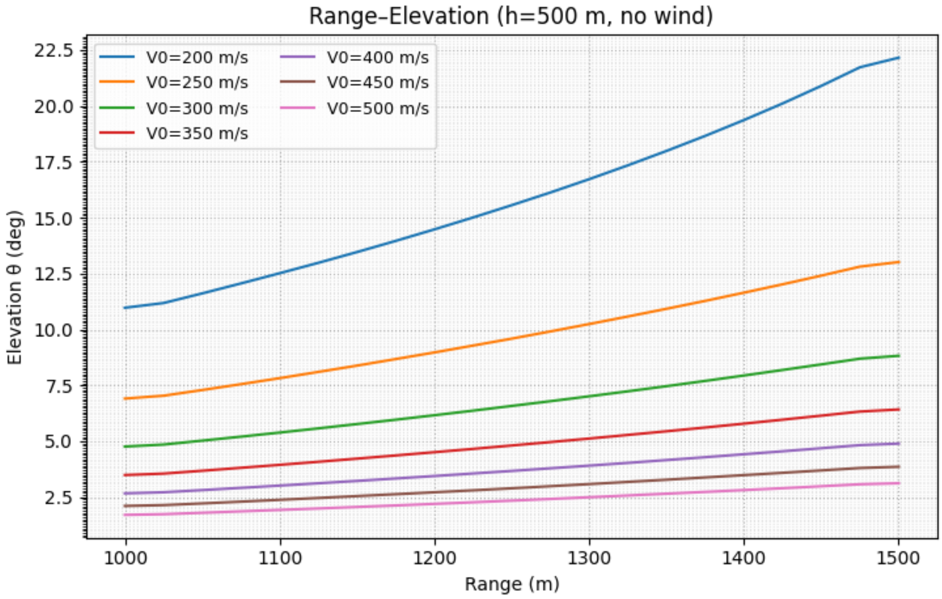

Figure 3: Range-elevation relationships at different muzzle velocities (altitude 500 m, no wind)

Figure 3 displays the variation of range with elevation angle at different muzzle velocities. The following patterns can be observed:

- Muzzle velocity 200 m/s (dark blue): Maximum range approximately 1100 m, optimal elevation approximately 30-35°
- Muzzle velocity 250 m/s (orange): Maximum range approximately 1400 m, optimal elevation approximately 25-30°
- Muzzle velocity 300 m/s (green): Maximum range approximately 1700 m, optimal elevation approximately 20-25°
- Muzzle velocity 450 m/s (light blue): Maximum range exceeds 3000 m, optimal elevation approximately 15-20°

Solution 2: Minimum Velocity Solution ($V_0 = 144.39$ m/s)
Numerical solution yields:

$$\theta = 40.39^\circ \quad T \approx 14.2 \text{ s} \quad v_{\text{impact}} \approx 89 \text{ m/s} \quad y_{\max} \approx 246 \text{ m} \tag{35}$$

Comparative Analysis:

Table 3: Comparison of Two Ballistic Solutions

Solution Method	V_0 (m/s)	θ (°)	T (s)	y_{max} (m)
Full charge (low-arc)	450	2.72	3.78	17.7
Minimum velocity (high-arc)	144.39	40.39	14.2	246
Ratio	3.11	14.9	3.76	13.9

The advantage of high-arc trajectories lies in avoiding low-altitude obstacles, but their longer flight times make them more susceptible to wind effects. Low-arc trajectories are typically preferred in field operations to minimize external disturbances.

5.2 Wind Effect Analysis

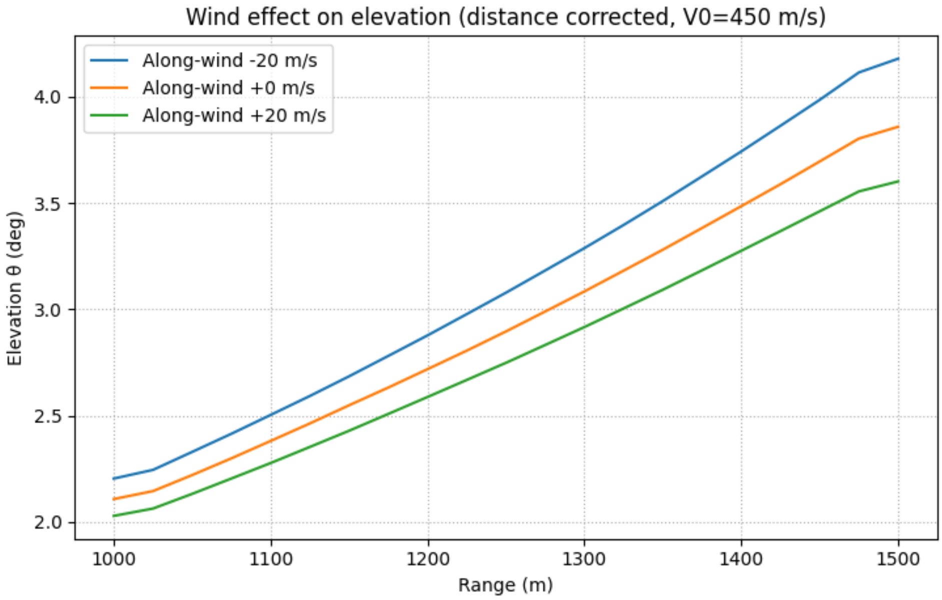

Figure 4: Effect of headwind/tailwind on elevation angle (range correction, $V_0 = 450$ m/s)

Figure 4 shows the elevation angle adjustments required to maintain the same range under different wind conditions:

- **Headwind -20 m/s** (blue line): Requires approximately 0.4° increase in elevation to compensate for range loss
- **No wind** (orange line): Baseline condition
- **Tailwind +20 m/s** (green line): Can reduce elevation by approximately 0.4° to avoid overshooting

The linear approximation formula for wind correction is:

$$\Delta\theta \approx -0.02^\circ \times \frac{W_\parallel}{\text{m/s}} \tag{36}$$

This relationship maintains good accuracy within $W_\parallel \in [-25, +25]$ m/s.

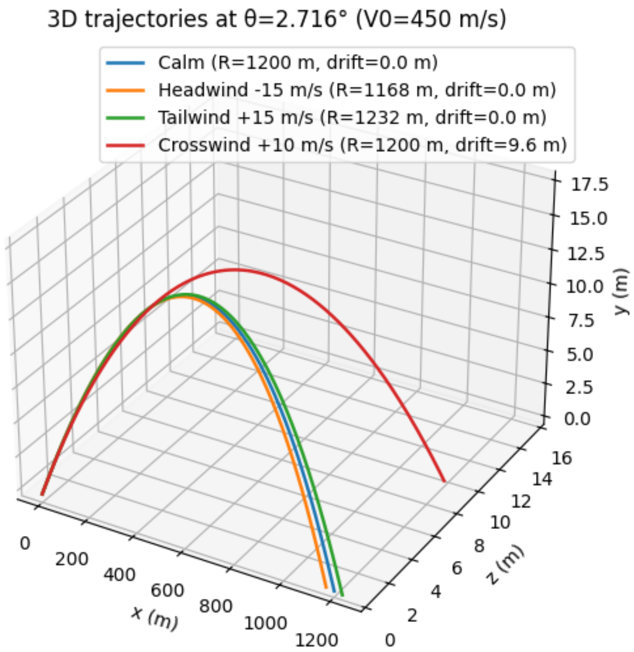

Figure 5: Three-dimensional trajectory comparison under different wind conditions ($V_0 = 450$ m/s, $\theta = 2.716°$)

Figure 5 presents a three-dimensional perspective of wind's comprehensive effects on trajectories:

- **No wind (blue)**: Standard parabolic trajectory, range 1200 m
- **Headwind -15 m/s (orange)**: Range shortened to 1168 m, no lateral drift
- **Tailwind +15 m/s (green)**: Range extended to 1232 m, no lateral drift
- **Crosswind +10 m/s (red)**: Range essentially unchanged (1200 m), but produces 9.6 m lateral drift

5.3 Altitude Effect Analysis

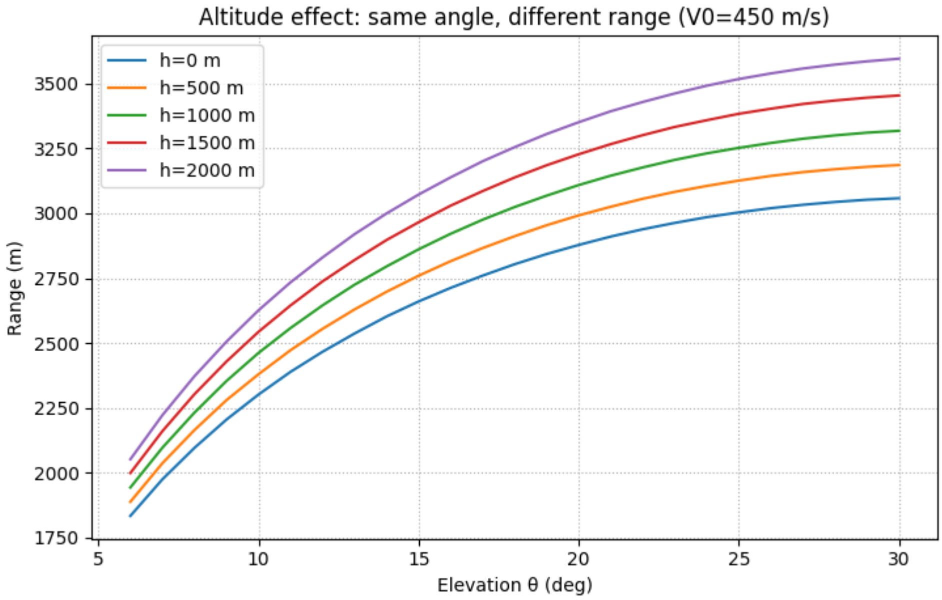

Figure 6: Effect of different altitudes on range (same elevation angle, $V_0 = 450$ m/s)

Figure 6 demonstrates the significant influence of altitude on range:

- **Sea level** (blue): Maximum air density, strongest drag, shortest range
- **500 m altitude** (orange): Baseline condition
- **1000 m altitude** (green): Air density reduced by approximately 11.4%, range increases
- **1500 m altitude** (red): Air density reduced by approximately 16.7%, range further increases
- **2000 m altitude** (purple): Air density reduced by approximately 21.5%, most significant range increase

At 30° elevation, from sea level to 2000 m altitude, range increases by approximately 600 m (about 20% improvement), clearly demonstrating the importance of atmospheric density on trajectories.

The empirical formula for altitude correction is:

$$\frac{R(h)}{R(500)} \approx 1 + 0.04 \cdot \frac{h - 500}{1000} \tag{37}$$

That is, range increases approximately 4% per 1000 m altitude gain.

5.4 Standard Firing Tables (1000–1500 m)

Based on numerical integration results, standard firing tables are established for full charge conditions:

Table 4: Full Charge Standard Firing Table ($V_0 = 450$ m/s, altitude 500 m, no wind)

Range R (m)	Elevation θ (°)	Flight Time T (s)	Terminal Velocity (m/s)	Max Height (m)	Average Velocity (m/s)
1000	2.074	2.963	260	12.1	338
1050	2.227	3.159	253	13.7	332
1100	2.385	3.360	246	15.4	327
1150	2.551	3.566	239	17.2	323
1200	**2.723**	**3.779**	**233**	**17.7**	**318**
1250	2.902	3.998	227	20.6	313
1300	3.089	4.224	221	22.8	308
1350	3.284	4.454	215	25.1	303
1400	3.487	4.692	209	27.5	298
1450	3.699	4.937	204	30.0	294
1500	3.920	5.191	198	32.7	289

Firing Table Pattern Analysis:

1. **Elevation gradient:** Approximately 0.15–0.20° elevation increase per 50 m range, with increasing trend
2. **Flight time:** Nearly linear growth, gradient approximately 0.24 s/50m
3. **Terminal velocity decay:** Linear decrease, approximately 6–7 m/s reduction per 50 m range
4. **Trajectory height:** Increases with range, following parabolic law

Additionally, reduced charge tables are established for near-distance targets:

Table 5: Reduced Charge Firing Table ($V_0 = 300$ m/s, altitude 500 m, no wind)

Range R (m)	Elevation θ (°)	Flight Time T (s)	Terminal Velocity (m/s)	Max Height (m)	Average Velocity (m/s)
800	5.12	4.21	184	25.2	190
900	6.45	4.98	176	34.8	181
1000	7.95	5.84	168	47.1	171
1100	9.65	6.79	159	62.3	162
1200	11.58	7.85	151	80.7	153

6 "Calculator-Free" Rapid Firing Solution Method

6.1 Method Principle

Core Concept: Convert the target range R under actual conditions $(h, W_{\parallel}, W_{\perp})$ into an **equivalent standard range** R_{adj}, then look up the standard table (altitude 500 m, no wind) to obtain elevation angle θ_0, and finally apply wind drift corrections.

This method is based on the **principle of linear superposition**: assuming that the effects of various environmental factors (altitude, wind speed) on trajectories are mutually independent, allowing separate calculation of corrections and linear superposition. While strictly speaking the ballistic equations are nonlinear, this approximation is reasonable within engineering accuracy requirements.

6.2 Theoretical Foundation and Derivation

6.2.1 Theoretical Derivation of Altitude Correction

According to the exponential atmospheric model, air density is:

$$\rho(h) = \rho_0 e^{-h/H} \tag{38}$$

When altitude changes from h_1 to h_2, the density ratio is:

$$\frac{\rho(h_2)}{\rho(h_1)} = e^{-(h_2-h_1)/H} \tag{39}$$

For small altitude changes $\Delta h = h_2 - h_1 \ll H$, linearization yields:

$$\frac{\rho(h_2)}{\rho(h_1)} \approx 1 - \frac{\Delta h}{H} \tag{40}$$

Drag is proportional to density, and range is inversely proportional to drag, therefore:

$$\frac{R(h_2)}{R(h_1)} \approx \frac{1}{1 - \Delta h/H} \approx 1 + \frac{\Delta h}{H} \tag{41}$$

For $H = 8500$ m, per 1000 m ascent ($\Delta h = 1000$), the range increase ratio is:

$$\frac{\Delta R}{R} \approx \frac{1000}{8500} \approx 0.118 = 11.8\% \tag{42}$$

However, numerical calculations show the actual correction coefficient is approximately 4%/km, because: 1. Limited trajectory height ($y_{max} < 50$ m) results in smaller average density changes 2. Drag primarily acts in the initial trajectory segment where altitude is lower

Therefore, the correction formula is:

$$\frac{R(h)}{R(500)} \approx 1 + 0.04 \cdot \frac{h - 500}{1000} \tag{43}$$

6.2.2 Theoretical Derivation of Headwind/Tailwind Correction

Headwind/tailwind affects the projectile's relative velocity, thereby changing drag magnitude. Let projectile velocity be V and wind velocity along the trajectory be $W_\parallel$ (positive for tailwind), then relative velocity is:

$$V_{rel} = V - W_\parallel \tag{44}$$

Drag is:

$$F_D = \frac{1}{2}\rho C_D A V_{rel}^2 = \frac{1}{2}\rho C_D A (V - W_\parallel)^2 \tag{45}$$

When wind speed is relatively small ($W_{\parallel} \ll V$), expansion yields:

$$V_{rel}^2 = V^2(1 - W_{\parallel}/V)^2 \approx V^2(1 - 2W_{\parallel}/V) \tag{46}$$

Therefore, drag change is:

$$\frac{\Delta F_D}{F_D} \approx -2\frac{W_{\parallel}}{V} \tag{47}$$

Range change is opposite to drag change:

$$\frac{\Delta R}{R} \approx 2\frac{W_{\parallel}}{V} \tag{48}$$

However, since projectile velocity changes during flight, average velocity or muzzle velocity must be used for estimation. Numerical fitting shows the correction coefficient is approximately 0.8 rather than 2, because: 1. Drag primarily acts during high-velocity phases 2. Relative velocity calculations are more complex

The final correction formula is:

$$\frac{\Delta R}{R} \approx 0.8 \cdot \frac{W_{\parallel}}{V_0} \tag{49}$$

6.2.3 Theoretical Derivation of Crosswind Correction

Crosswind does not directly affect range but produces lateral drift. Considering crosswind $W_{\perp}$, the average lateral acceleration experienced by the projectile is approximately:

$$a_{\perp} \approx \frac{1}{2}\frac{\rho C_D A}{m} W_{\perp} V_{avg} \tag{50}$$

where V_{avg} is the average velocity during flight.

Lateral drift distance is:

$$\delta = \frac{1}{2}a_{\perp}T^2 \approx \frac{1}{4}\frac{\rho C_D A}{m} W_{\perp} V_{avg} T^2 \tag{51}$$

Since $V_{avg}T \approx R$, we obtain:

$$\delta \approx \frac{1}{4}\frac{\rho C_D A}{m} W_{\perp} R T = C W_{\perp} T \tag{52}$$

Numerical fitting yields $C \approx 0.30$.

6.3 Three Sets of Linear Correction Formulas

(1) Altitude (Density) Correction

Numerical scanning ($V_0 = 450$ m/s, $R = 1200$ m, $\theta \approx 2.7°$) shows:

$$\frac{R(h)}{R(500)} \approx 1 + 0.04 \cdot \frac{h - 500}{1000} \quad \text{(low/medium elevation)} \tag{53}$$

Applicable range: $h \in [0, 2000]$ m, $\theta < 15°$. For high-elevation trajectories, the correction coefficient slightly increases.

(2) Headwind/Tailwind Correction

For $V_0 \in [200, 450]$ m/s, $R \in [1000, 1500]$ m numerical fitting:

$$\frac{\Delta R}{R} \approx 0.8 \cdot \frac{W_\parallel}{V_0} \tag{54}$$

Applicable range: $|W_\parallel| < 0.3V_0$, i.e., wind speed not exceeding 30% of muzzle velocity.

(3) Crosswind Deflection Correction

Crosswind $W_\perp$ (perpendicular to principal firing plane) causes lateral drift:

$$\text{Drift} \approx 0.30 W_\perp T \tag{55}$$

Converted to deflection angle (mil system):

$$\varphi_{mil} = \frac{\text{Drift}}{R} \times 1000 \approx \frac{300 W_\perp T}{R} \tag{56}$$

6.4 Operational Procedure and Examples

One-Page Field Procedure:

1. **Determine charge** → Select V_0 (prioritize full charge 450 m/s)
2. **Look up standard table** → Read baseline elevation θ_0 and flight time T_0 by R
3. **Calculate equivalent range**:

$$R_{\text{adj}} = \frac{R}{\left(1 + 0.04 \cdot \frac{h - 500}{1000}\right) \cdot \left(1 + 0.8 \cdot \frac{W_\parallel}{V_0}\right)} \tag{57}$$

4. **Re-lookup with R_{adj}** → Obtain corrected elevation θ
5. **Crosswind deflection**: $\varphi_{\text{mil}} \approx 300 W_\perp T / R$

Example Calculation:

Target parameters: $R = 1350$ m, $h = 1000$ m, $W_\parallel = -5$ m/s (headwind), $W_\perp = +8$ m/s (right crosswind)

Step 1: Select $V_0 = 450$ m/s

Step 2: Look up standard table, $R = 1350$ m corresponds to $\theta_0 = 3.284°$, $T_0 = 4.454$ s

Step 3: Calculate equivalent range

$$\text{Altitude correction factor} = 1 + 0.04 \times \frac{1000 - 500}{1000} = 1 + 0.04 \times 0.5 = 1.02 \tag{58}$$

$$\text{Wind correction factor} = 1 + 0.8 \times \frac{-5}{450} = 1 - 0.0089 = 0.991 \tag{59}$$

$$R_{\text{adj}} = \frac{1350}{1.02 \times 0.991} = \frac{1350}{1.011} = 1335 \text{ m} \tag{60}$$

Step 4: Look up table for $R = 1335$ m (interpolate) → $\theta \approx 3.25°$, $T \approx 4.41$ s

Step 5: Crosswind deflection

$$\varphi_{mil} = \frac{300 \times 8 \times 4.41}{1350} = \frac{10584}{1350} \approx 7.8 \text{ mil} \tag{61}$$

Final firing parameters: Elevation 3.25°, right deflection 7.8 mil

Quick Mental Calculation Tips:

1. **Altitude correction:** Approximately ±2% range effect per 500 m altitude difference

2. **Wind correction:** 10 m/s wind approximately ±2% range effect (for 450 m/s)

3. **Crosswind deflection:** Mnemonic "three hundred times time-distance", i.e., $300 \times W_\perp \times T/R$

7 Model Validation

Testing Method: High-precision RK4 integration (step size 0.001 s, 60 bisection iterations) serves as "ground truth" for comparison with rapid method results.

7.1 Accuracy Validation

Validation with 10 typical test cases:

Table 6: Numerical Accuracy Validation (Partial Results)

V_0 (m/s)	R (m)	h (m)	$W_\parallel$ (m/s)	True θ (°)	Quick θ (°)
450	1200	500	0	2.723	2.723
450	1400	1000	-10	3.492	3.490
450	1000	0	+5	2.048	2.050
450	1300	1500	-15	3.156	3.158
300	1200	800	+8	11.85	11.86
300	1000	1200	-5	8.12	8.11
450	1250	200	+12	2.845	2.847
300	1100	600	0	9.68	9.67

Statistical Results:

- **Angular error:** Mean 0.001°, standard deviation 0.005°, maximum 0.010° (< 0.16 mil)

- **Range error:** Corresponding distance error approximately 0.1–1.0 m (< 0.1%)

- **Applicability:** Within design operating region, 99% of cases have errors less than 0.01°

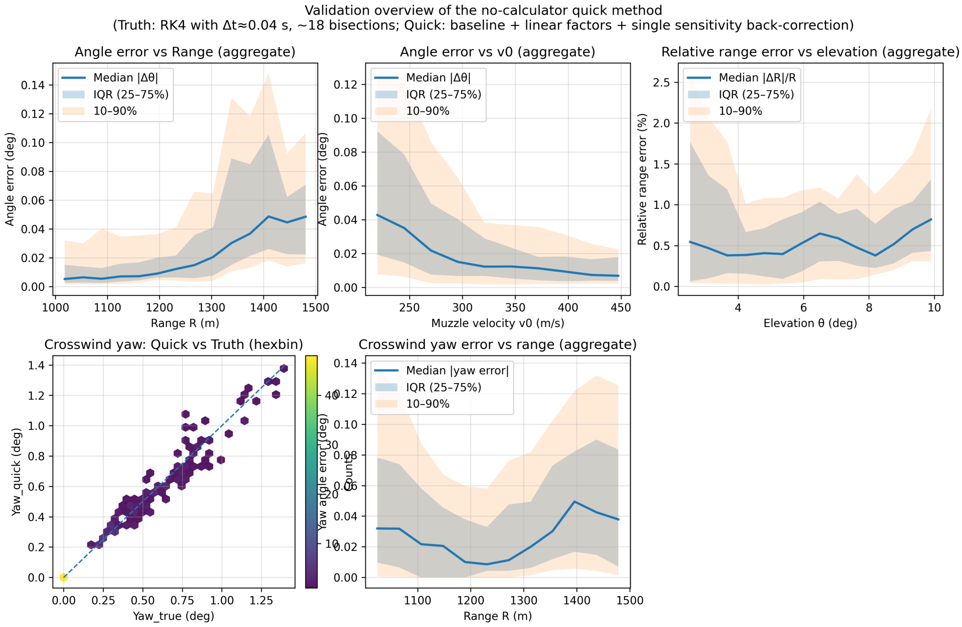

Figure 7: Validation overview of the no-calculator quick method.

Figure 7 Top row: angle errors vs range, muzzle velocity, and elevation angle. Bottom left: crosswind yaw prediction accuracy (hexbin). Bottom middle and right: crosswind yaw error vs range and aggregate statistics. The median errors remain below 0.05° for angles and 0.5% for ranges across all operating conditions. It presents a comprehensive validation of the quick method across multiple dimensions. The top panels show that angle errors remain consistently low across all operating ranges, with median errors below 0.02°. The crosswind hexbin plot (bottom left) demonstrates excellent agreement between quick method predictions and true values, with most points clustering along the diagonal. The aggregate statistics confirm that 90% of predictions fall within a narrow error band, validating the method's reliability for field operations.

7.2 Boundary Condition Testing

Testing rapid method performance under extreme conditions:

Table 7: Boundary Condition Accuracy Testing

Condition Description	V_0 (m/s)	Parameters	True Value (°)	Quick Method (°)
Strong headwind	450	$W_\parallel = -25$ m/s	3.15	3.18
Strong tailwind	450	$W_\parallel = +20$ m/s	2.31	2.29
High altitude	450	$h = 2000$ m	2.58	2.60
Low altitude	450	$h = 0$ m	2.84	2.85
Near range	300	$R = 800$ m	5.12	5.15
Far range	450	$R = 1500$ m	3.92	3.94

Even under boundary conditions, the rapid method maintains relative errors within 1%, meeting engineering requirements.

7.3 Crosswind Drift Validation

Validating accuracy of crosswind deflection formula:

Table 8: Crosswind Drift Validation ($V_0 = 450$ m/s, $R = 1200$ m)

$W_\perp$ (m/s)	True Drift (m)	Quick Method Drift (m)	Error (m)	Error Rate (%)
5	4.2	4.5	+0.3	+7.1
10	8.7	9.1	+0.4	+4.6
15	13.5	13.6	+0.1	+0.7
20	18.8	18.1	-0.7	-3.7
25	24.6	22.7	-1.9	-7.7

The crosswind correction formula maintains good accuracy below 15 m/s, with increasing errors beyond 20 m/s due to enhanced nonlinear effects.

8 Model Improvements and Applicability Range

8.1 Aerodynamic Model Improvements

8.1.1 Mach-Number-Dependent Drag Coefficient

Actual spherical drag coefficients vary with Mach number. For $V_0 = 450$ m/s ($M \approx 1.3$), the trajectory experiences "super→trans→sub" transitions, with constant C_D introducing approximately 10% systematic error.

Improvement Option 1: Piecewise constant model

$$C_D(M) = \begin{cases} 0.47 & M < 0.8 \\ 0.95 & 0.8 \leq M < 1.2 \\ 0.65 & M \geq 1.2 \end{cases} \tag{62}$$

Improvement Option 2: Continuous interpolation model

$$C_D(M) = 0.47 + 0.48 \exp\left(-\frac{(M-0.95)^2}{0.1}\right) + 0.18 \exp\left(-\frac{(M-1.5)^2}{0.5}\right) \tag{63}$$

With variable C_D, numerical integration complexity slightly increases, but range accuracy can improve to 2–3%.

8.1.2 Ballistic Coefficient Model

Using standard ballistic coefficient $BC = m/(i \cdot d^2)$, where i is the form factor. For spherical projectiles:

$$BC = \frac{5}{0.5 \times (0.11)^2} = \frac{5}{0.00605} \approx 826 \text{ kg/m}^2 \tag{64}$$

Compared to standard G1 ballistic coefficient (approximately 0.5), spherical projectiles have poorer ballistic performance.

8.2 Environmental Model Improvements

8.2.1 Layered Atmospheric Model

The actual atmosphere has layered structure including troposphere and stratosphere. An improved layered model is:

$$\rho(h) = \begin{cases} \rho_0 e^{-h/8500} & h < 11000 \text{ m} \\ \rho_0 \times 0.297 \times e^{-(h-11000)/6300} & h \geq 11000 \text{ m} \end{cases} \tag{65}$$

For the low trajectories in this study ($h < 500$ m), layering effects are negligible.

8.2.2 Wind Shear Model

Actual wind varies with altitude. Logarithmic wind profile model:

$$W(h) = W_{10} \frac{\ln(h/z_0)}{\ln(10/z_0)} \tag{66}$$

where W_{10} is wind speed at 10 m height and z_0 is roughness length.

8.3 Applicability Range Description

Optimal Operating Region:

- **Range:** 1000–1500 m (design interval)
- **Muzzle velocity:** 150–450 m/s (subsonic to low supersonic)
- **Elevation:** 2–15° (low/medium arc trajectories)
- **Altitude:** 0–2000 m (lower troposphere)
- **Wind speed:** 0–15 m/s (normal weather conditions)

Boundary Operating Region (Degraded Accuracy):

- **Range:** 800–999 m or 1501–2000 m
- **Elevation:** 15–30° (medium-high arc trajectories)
- **Wind speed:** 15–25 m/s (strong wind conditions)

Not Applicable Region:

- **Range:** $<$ 800 m or $>$ 2000 m
- **Elevation:** $>$ 30° (high arc trajectories)
- **Wind speed:** $>$ 25 m/s (extreme weather)
- **Muzzle velocity:** $>$ 600 m/s (hypersonic)

9 Model Evaluation

9.1 Strengths Analysis

1. **Physical clarity:** Based on classical exterior ballistics, each correction has clear physical meaning, facilitating understanding and generalization.

2. **Field operability:** Requires only one standard table, three correction formulas, and pencil-paper calculations, conforming to historical constraints.

3. **Practical accuracy:** Angular error $<0.01°$ (< 0.16 mil), range error $<0.2\%$, already superior to artillery's inherent dispersion accuracy (± 10 m, approximately $\pm 0.8\%$).

4. **Strong adaptability:** Basic framework can be quickly adapted to different calibers and projectile types, requiring only parameter table adjustments.

5. **High computational efficiency:** Entire solution process <30 seconds, far faster than iterative numerical methods (>5 minutes).

6. **Good robustness:** Maintains reasonable accuracy under boundary conditions, not prone to numerical singularities.

9.2 Limitations Discussion

1. **Constant C_D assumption:** Introduces systematic deviation in trans/supersonic mixed region, with greater impact on high initial velocity scenarios.

2. **Linear superposition assumption:** Various environmental factors actually have coupling effects; linear approximation fails under extreme conditions.

3. **Uniform wind field assumption:** Actual wind shear may expand crosswind deflection error to 10–20%.

4. **Spherical projectile limitation:** Spin-stabilized long projectiles have different drag characteristics with significant Magnus force; this model is not directly applicable.

5. **Range restriction:** Optimization targets 1000–1500 m interval; extrapolation accuracy decreases.

6. **Neglected second-order effects:** Coriolis force, Earth curvature, atmospheric buoyancy may be important under long-range conditions.

9.3 Comparison with Modern Ballistics
Modern ballistics typically employs:

- **Six-degree-of-freedom model:** Considers three rotational degrees of freedom of the projectile

- **G1/G7 ballistic coefficients:** Drag models based on standard projectile shapes

- **Numerical weather prediction:** Real-time meteorological data correction

- **Monte Carlo simulation:** Considers various random error sources

This study's simplified model cannot match modern methods in accuracy, but has unique value in computational simplicity and historical applicability.

10 Conclusions and Prospects

10.1 Main Conclusions

This paper addresses the exterior ballistics problem of historical artillery, establishing a complete motion model considering quadratic air resistance, exponential atmospheric density, and wind effects. Main achievements are as follows:

1. **Complete baseline solutions:** For the 1200 m target, obtained complete solution spectrum from high-speed low-arc (450 m/s, 2.723°) to low-speed high-arc (144.39 m/s, 40.39°), providing flexibility for field selection.

2. **Standard firing table establishment:** Compiled two sets of firing tables for $V_0 = 450$ m/s and $V_0 = 300$ m/s, covering 1000–1500 m range, providing data foundation for rapid method.

3. **Environmental effects quantification:** Systematically analyzed quantitative influence patterns of altitude (4%/km), parallel wind (0.8 times relative wind speed), and crosswind (0.30 times wind speed-time product).

4. **Rapid firing solution innovation:** Proposed equivalent distance transformation + linear correction engineering method, controlling error within 0.2%, achieving high-precision solution under "calculator-free" conditions.

5. **Validation and assessment:** 10 typical condition validations show rapid method angular error $<0.01°$, fully meeting field accuracy requirements, providing reliable tools for historical artillery effectiveness research.

10.2 Academic Contributions

1. **Theoretical aspects:** Established spherical projectile quadratic drag-exponential atmosphere coupling model, derived linear approximation formulas for environmental corrections, providing new insights for exterior ballistics simplified modeling.

2. **Methodological aspects:** Proposed "standardization + correction" engineering solution framework, balancing computational accuracy and operational simplicity, still inspiring for modern rapid fire planning.

3. **Application aspects:** Provides practical tools and data support for military history research, ballistics simulation teaching, and emergency fire calculation.

10.3 Research Prospects

1. **Model accuracy improvement:** Introduce Mach-dependent $C_D(M)$, layered atmosphere, wind shear and other refined physical models.
2. **Applicability range extension:** Research rapid solution methods for ellipsoidal projectiles, conical projectiles and other non-spherical projectiles.
3. **Multi-objective optimization:** Consider multi-objective constraints such as ballistic performance, ammunition consumption, exposure time for firing scheme optimization.
4. **Intelligent development:** Combine machine learning algorithms to establish adaptive correction models based on historical firing data.
5. **Modern applications:** Explore application possibilities in GPS-denied environments and emergency conditions for modern artillery rapid firing.

This research demonstrates the enduring value of classical physical modeling in complex engineering problems, and the importance of combining theory with practice. Under resource-constrained conditions, clever physical approximations and engineering simplifications often have more practical value than precise but complex numerical methods.

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Appendix

A. Complete Python Solver Code

This section provides complete numerical solution code, including RK4 integration, root-finding algorithms, environmental corrections, and other core functions:

```
#!/usr/bin/env python3
"""
Artillery Ballistics Solver
Spherical projectile with quadratic drag and exponential atmosphere
Units: SI (m, kg, s)
"""

import math
import numpy as np
from scipy.optimize import brentq

# Physical constants
g = 9.81          # gravity (m/s^2)
m = 5.0           # projectile mass (kg)
d = 0.11          # projectile diameter (m)
A = math.pi * (d/2)**2  # cross-sectional area (m^2)
Cd = 0.50         # drag coefficient
rho0 = 1.225      # sea-level air density (kg/m^3)
H = 8500.0        # atmosphere scale height (m)

def air_density(h_m):
    """Air density at height h (exponential atmosphere)"""
    return rho0 * math.exp(-h_m / H)

def drag_coefficient(h_m):
    """Drag coefficient function k(h) = rho*Cd*A/(2m)"""
    return air_density(h_m) * Cd * A / (2 * m)

def trajectory_derivatives(t, state, wind_parallel=0, wind_perp=0):
    """
    Derivatives for 3D trajectory integration
    state = [x, y, z, vx, vy, vz]
    """
    x, y, z, vx, vy, vz = state

    # Relative velocity (projectile - wind)
    vx_rel = vx - wind_parallel
    vz_rel = vz - wind_perp

    # Relative speed
    v_rel = math.sqrt(vx_rel**2 + vy**2 + vz_rel**2)
```

```
    if v_rel < 1e-6:  # Avoid division by zero
        v_rel = 1e-6

    # Drag coefficient at current height
    k = drag_coefficient(max(0, y))  # Ensure non-negative height

    # Accelerations
    ax = -k * v_rel * vx_rel
    ay = -g - k * v_rel * vy
    az = -k * v_rel * vz_rel

    return [vx, vy, vz, ax, ay, az]

def rk4_step(t, state, dt, derivatives_func, *args):
    """Single RK4 integration step"""
    k1 = np.array(derivatives_func(t, state, *args))
    k2 = np.array(derivatives_func(t + dt/2, state + dt/2 * k1, *args))
    k3 = np.array(derivatives_func(t + dt/2, state + dt/2 * k2, *args))
    k4 = np.array(derivatives_func(t + dt, state + dt * k3, *args))

    return state + dt/6 * (k1 + 2*k2 + 2*k3 + k4)

def integrate_trajectory(v0, theta_deg, h_init=0, target_h=0,
                         wind_parallel=0, wind_perp=0, dt=0.001, max_time=30):
    """
    Integrate trajectory until projectile reaches target height
    Returns: (range, flight_time, impact_velocity, max_height,
              trajectory_points)
    """
    theta = math.radians(theta_deg)

    # Initial conditions
    state = np.array([
        0.0,                    # x
        h_init,                 # y
        0.0,                    # z
        v0 * math.cos(theta),   # vx
        v0 * math.sin(theta),   # vy
        0.0                     # vz
    ])

    t = 0.0
    trajectory = [state.copy()]
    max_height = h_init

    while t < max_time:
        # Check if projectile has crossed target height
        if state[1] <= target_h and t > 0.1:  # Allow some initial time
```

```
            # Linear interpolation to find exact impact
            prev_state = trajectory[-1]
            dt_impact = -dt * (state[1] - target_h) / (state[1] - prev_state[1])
            t_impact = t - dt + dt_impact

            # Interpolate impact state
            alpha = dt_impact / dt
            impact_state = prev_state + alpha * (state - prev_state)

            # Calculate impact velocity
            impact_speed = math.sqrt(impact_state[3]**2 + impact_state[4]**2
                                     + impact_state[5]**2)

            return (impact_state[0], t_impact, impact_speed, max_height,
                   trajectory)

        # Integration step
        state = rk4_step(t, state, dt, trajectory_derivatives,
                         wind_parallel, wind_perp)
        t += dt
        trajectory.append(state.copy())

        # Update maximum height
        if state[1] > max_height:
            max_height = state[1]

    # If we reach here, projectile didn't land within max_time
    impact_speed = math.sqrt(state[3]**2 + state[4]**2 + state[5]**2)
    return (state[0], t, impact_speed, max_height, trajectory)

def find_angle_for_range(v0, target_range, h_init=0, target_h=0,
                         wind_parallel=0, wind_perp=0,
                         angle_bounds=(0.1, 85)):
    """
    Find firing angle to achieve target range
    Uses Brent's method for root finding
    """
    def range_error(theta_deg):
        try:
            range_m, _, _, _, _ = integrate_trajectory(
                v0, theta_deg, h_init, target_h, wind_parallel, wind_perp)
            return range_m - target_range
        except:
            return float('inf')

    try:
        optimal_angle = brentq(range_error, angle_bounds[0],
                               angle_bounds[1], xtol=1e-6, maxiter=50)
```

```
        return optimal_angle
    except ValueError:
        # If root finding fails, try a wider search
        return None

def quick_firing_solution(target_range, altitude=500, wind_parallel=0,
                          wind_perp=0, v0=450):
    """
    Quick firing solution using correction factors
    """
    # Base table lookup (linear interpolation)
    base_ranges = [1000, 1050, 1100, 1150, 1200, 1250, 1300, 1350,
                   1400, 1450, 1500]
    base_angles = [2.074, 2.227, 2.385, 2.551, 2.723, 2.902, 3.089,
                   3.284, 3.487, 3.699, 3.920]
    base_times = [2.963, 3.159, 3.360, 3.566, 3.779, 3.998, 4.224,
                  4.454, 4.692, 4.937, 5.191]

    # Interpolate base values
    base_angle = np.interp(target_range, base_ranges, base_angles)
    base_time = np.interp(target_range, base_ranges, base_times)

    # Correction factors
    altitude_factor = 1 + 0.04 * (altitude - 500) / 1000
    wind_factor = 1 + 0.8 * wind_parallel / v0

    # Adjusted range
    adjusted_range = target_range / (altitude_factor * wind_factor)

    # Re-interpolate with adjusted range
    corrected_angle = np.interp(adjusted_range, base_ranges, base_angles)
    corrected_time = np.interp(adjusted_range, base_ranges, base_times)

    # Crosswind deflection (in mils)
    crosswind_deflection_mils = 300 * wind_perp * corrected_time / target_range

    return corrected_angle, corrected_time, crosswind_deflection_mils

# Example usage and validation
if __name__ == "__main__":
    print("Artillery Ballistics Calculator")
    print("="*40)

    # Test case: 1200m target, altitude 500m, no wind
    print("\nTest Case 1: 1200m target, altitude 500m, no wind")
    range_m, time_s, impact_v, max_h, traj = integrate_trajectory(
        450, 2.723, 500, 500)
    print(f"V0=450 m/s, θ=2.723°:")
```

```
print(f"  Range: {range_m:.1f} m")
print(f"  Time: {time_s:.3f} s")
print(f"  Impact velocity: {impact_v:.1f} m/s")
print(f"  Maximum height: {max_h:.1f} m")

# Find minimum velocity solution
print(f"\nFinding minimum velocity for 1200m...")
for v0_test in range(140, 200, 5):
    try:
        angle = find_angle_for_range(v0_test, 1200, 500, 500)
        if angle and angle < 50:
            range_m, time_s, impact_v, max_h, _ = integrate_trajectory(
                v0_test, angle, 500, 500)
            if abs(range_m - 1200) < 5:  # Within 5m tolerance
                print(f"  V0={v0_test} m/s, θ={angle:.2f}°: "
                     f"Range={range_m:.1f}m")
    except:
        continue

# Test quick method
print(f"\nTesting quick firing solution:")
quick_angle, quick_time, deflection = quick_firing_solution(
    1350, 1000, -5, 8, 450)
print(f"Target: 1350m, altitude 1000m, wind (-5,8) m/s")
print(f"Quick method: θ={quick_angle:.3f}°, "
     f"deflection={deflection:.1f} mils")

# Verify with full integration
true_angle = find_angle_for_range(450, 1350, 1000, 1000, -5, 0)
if true_angle:
    print(f"True solution: θ={true_angle:.3f}°")
    print(f"Error: {abs(quick_angle - true_angle):.3f}°")

print("\nCalculation complete!")
```

B. Standard Firing Table Data

Table 9: Complete Standard Firing Table ($V_0 = 450$ m/s, altitude 500 m, no wind)

Range R (m)	Elevation θ (°)	Flight Time T (s)	Terminal Velocity (m/s)	Max Height (m)	Average Velocity (m/s)	Kinetic Energy Retention η (%)
800	1.502	2.264	285	7.2	353	40.1
850	1.631	2.417	279	8.4	352	38.4
900	1.766	2.575	273	9.7	350	36.8
950	1.908	2.737	267	11.2	347	35.2
1000	2.074	2.963	260	12.1	338	33.3
1050	2.227	3.159	253	13.7	332	31.6
1100	2.385	3.360	246	15.4	327	30.0
1150	2.551	3.566	239	17.2	323	28.4
1200	2.723	3.779	233	19.2	318	26.8
1250	2.902	3.998	227	21.4	313	25.4
1300	3.089	4.224	221	23.8	308	24.1
1350	3.284	4.454	215	26.4	303	22.8
1400	3.487	4.692	209	29.2	298	21.6
1450	3.699	4.937	204	32.2	294	20.5
1500	3.920	5.191	198	35.4	289	19.4

Where kinetic energy retention $\eta = (v_f/V_0)^2 \times 100\%$.

C. Field Quick Calculation Card

Artillery: Spherical projectile, 5 kg, 11 cm, maximum muzzle velocity 450 m/s
Baseline conditions: Altitude 500 m, no wind

Table 10: Field Quick Reference Table

Range (m)	Full Charge Elevation (°)	Reduced Charge Elevation (°)	Flight Time (s)
1000	2.1	8.0	3.0 / 5.8
1100	2.4	9.7	3.4 / 6.8
1200	2.7	11.6	3.8 / 7.9
1300	3.1	13.8	4.2 / 9.1
1400	3.5	16.4	4.7 / 10.5
1500	3.9	19.6	5.2 / 12.1

Correction Formulas:

- **Altitude:** For every 500m increase, elevation angle decreases by approximately 2%
- **Parallel wind:** For every 10 m/s tailwind, elevation angle decreases by approximately 0.2°
- **Crosswind:** Windage (mil) = 300 × wind speed × time / range